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Derivations on Dual Triangular Banach Algebras

Ahmad Minapoor*

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Abstract Ideal Connes-amenability of dual Banach algebras was investigated in [17] by A. Minapoor, A. Bodaghi and D. Ebrahimi Bagha. They studied weak*-continuous derivations from dual Banach algebras into their weak*-closed two- sided ideals. This work considers weak*-continuous derivations of dual triangular Banach algebras into their weak*-closed two- sided ideals . We investigate when weak*-continuous derivations from these algebras into their weak*-closed ideals are inner?

Keywords Connes-amenable · Derivation · triangular Banach algebra

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1 Introduction

Let A be a Banach algebra and X be a Banach A-bimodule. Then a linear map $D : A \longrightarrow X$ is a derivation if

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for every $a, b \in A$. Let $x \in X$, and set $\delta_x(a) = a \cdot x - x \cdot a$ for every $a \in A$. Then δ_x is a derivation, these derivations are inner derivations. The space of continuous derivations from A into X is denoted by $\mathcal{Z}^1(A, X)$ and the subspace consisting of the inner derivations is $\mathcal{N}^1(A, X)$, the first cohomology group of A with coefficients in X is $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$.

*Corresponding author
Ahmad Minapoor
Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.
Tel.: +98-66-42468320
Fax: +98-66-42468320
E-mail: shp_np@yahoo.com

Let X^* be the dual Banach space of X. Then X^* is also a Banach A-bimodule by the following actions

$$\langle x, a \cdot f\rangle = \langle x \cdot a, f\rangle, \quad \text{and} \quad \langle x, f \cdot a\rangle = \langle a \cdot x, f\rangle,$$

for every $a \in A, x \in X$ and $f \in X^*$. The Banach algebra A is amenable if $\mathcal{H}^1(A, X^*) = \{0\}$ for each Banach A-bimodule X. Amenability of Banach algebras was introduced by Johnson in [11], where it is proved that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is an amenable group. Studying on amenability of C^*-algebras led to the new definition namely Connes-amenability. This notion defined on dual Banach algebras [13]. Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule X is called dual if there is a closed submodule X_* of X^* such that $X = (X_*)^*$ (X_* is called the predual of X). A Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule.

Let $\mathcal{A}$ be a dual Banach algebra. A dual Banach $\mathcal{A}$-bimodule X is called normal, if for every $x \in X$, the maps

$$\mathcal{A} \longrightarrow X, \quad a \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are *weak***-continuous* (w^*-continuous). Dual Banach algebra $\mathcal{A}$ is called Connes-amenable, if for every dual Banach $\mathcal{A}$-bimodule X, every w^*-continuous derivation $D : \mathcal{A} \longrightarrow X$ is inner, or equivalently, $\mathcal{H}^1_{w^*}(\mathcal{A}, X) = \{0\}$ [13]. Let I be a w^*-closed two sided ideal of $\mathcal{A}$ if $\mathcal{H}^1_{w^*}(\mathcal{A}, I) = \{0\}$ then $\mathcal{A}$ is called I-connes amenable. If $\mathcal{A}$ is I-connes amenable for every w^*-closed two sided ideal I of $\mathcal{A}$ then $\mathcal{A}$ is called ideally connes amenable [17]. Weak amenability of module extensions of Banach algebras studied by Zhang in [15], Forrest and Marcoux studied derivation of triangular Banach algebras in [7]. First Hochschild cohomology group of triangular Banach algebras studied in [8, **?**]. Connes-amenability of dual of module extensions of Banach algebras investigated in [5].

In [18] dual triangular Banach algebras were introduced and investigated w^*-continuous derivations from these algebras into themselves.

In this paper we study weak*-continuous derivations of dual triangular Banach algebras into their weak*-closed ideals. In a simillar manner of ideal Connes-amenability of dual Banach algebra that is defined in [17], we study some notes on connes- amenability of dual triangular Banach algebras with respect to their w^*-closed ideals.

2 Connes-amenability of module extensions of Banach algebras

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. The Banach algebra $\mathcal{A} \oplus_\infty X$ is defined in [5] with the algebra product,

$$(a, x)(b, y) = (ab, ay + xb)$$

and with the norm,

$$\|(a,x)\| = \max\{\|x\|, \|a\|\} \quad (a \in \mathcal{A}, x \in X).$$

Theorem 1 *[18] $\mathcal{A} \oplus_\infty X$ is Connes-amenable if and only if $\mathcal{A}$ is Connes-amenable and $X = 0$.*

Theorem 2 *[18] Let $\mathcal{A}$ and $\mathcal{B}$ be two dual Banach algebras and $\mathcal{M}$ be a dual Banach space that is a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. Then*

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix} = \mathcal{A} \oplus_\infty \mathcal{M} \oplus_\infty \mathcal{B},$$

with the sum and product being giving by the usual 2×2 matrix operations and internal module actions is an algebra. Furthermore, by the following norm is a Banach algebra:

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|(a,m,b)\| = \max\{\|a\|_{\mathcal{A}}, \|m\|_{\mathcal{M}}, \|b\|_{\mathcal{B}}\}.$$

Let X is a normal Banach $\mathcal{T}$-bimodule, it is also acted on $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$ from the left and from the right via the following actions:

$$x \cdot (a,m,b) = x \cdot a + x \cdot m + x \cdot b, \ (a,m,b) \cdot x = a \cdot x + m \cdot x + b \cdot x,$$

for every $a \in \mathcal{A}$, $m \in \mathcal{M}$ and $b \in \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital ($e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively) then $\mathcal{T}$ is unital with identity $(e_{\mathcal{A}}, 0, e_{\mathcal{B}}) = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$. Therefore, X become a unital $\mathcal{T}$-bimodule and we have

$$\begin{aligned} X &= e_{\mathcal{A}} \cdot X \cdot e_{\mathcal{A}} + e_{\mathcal{A}} \cdot X \cdot e_{\mathcal{B}} + e_{\mathcal{B}} \cdot X \cdot e_{\mathcal{A}} + e_{\mathcal{B}} \cdot X \cdot e_{\mathcal{B}} \\ &\quad + e_{\mathcal{A}} \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) + e_{\mathcal{B}} \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \\ &\quad + (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot e_{\mathcal{A}} + (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot e_{\mathcal{B}} \\ &\quad + (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}}). \end{aligned} \tag{1}$$

Note that action on the left on $(1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot e_{\mathcal{A}}$, $(1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot e_{\mathcal{B}}$ and $(1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}})$ is zero and action on the right on $e_{\mathcal{A}} \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}})$, $e_{\mathcal{B}} \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}})$ and $(1-e_{\mathcal{A}})(1-e_{\mathcal{B}}) \cdot X \cdot (1-e_{\mathcal{A}})(1-e_{\mathcal{B}})$ is zero. We use these notations in this paper: $X_{\mathcal{A}\mathcal{A}} = e_{\mathcal{A}} \cdot X \cdot e_{\mathcal{A}}, X_{\mathcal{B}\mathcal{B}} = e_{\mathcal{B}} \cdot X \cdot e_{\mathcal{B}}, X_{\mathcal{A}\mathcal{B}} = e_{\mathcal{A}} \cdot X \cdot e_{\mathcal{B}}$, and $X_{\mathcal{B}\mathcal{A}} = e_{\mathcal{B}} \cdot X \cdot e_{\mathcal{A}}$. If replace $\mathcal{T}$ instead of X, we have $X_{\mathcal{A}\mathcal{A}} = \mathcal{A}, X_{\mathcal{B}\mathcal{B}} = \mathcal{B}, X_{\mathcal{A}\mathcal{B}} = \mathcal{M}$, and $X_{\mathcal{B}\mathcal{A}} = 0$.

Suppose that $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are w^*-closed two sided ideals of $\mathcal{A}$ and $\mathcal{B}$ respectively, and let Y be a dual $\mathcal{A}-\mathcal{B}$-submodule of M such that $I_{\mathcal{A}}M \cup MI_{\mathcal{B}} \subset Y$, then It is easy to show that $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & Y \\ & I_{\mathcal{B}} \end{bmatrix}$ is a w^*-closed two sided ideal in $\mathcal{T}$. If replace $I_{\mathcal{T}}$ instead of X, we have $X_{\mathcal{A}\mathcal{A}} = I_{\mathcal{A}}, X_{\mathcal{B}\mathcal{B}} = I_{\mathcal{B}}, X_{\mathcal{A}\mathcal{B}} = Y$, and $X_{\mathcal{B}\mathcal{A}} = 0$.

Lemma 1 *Let $\mathcal{T}$ be a dual triangular Banach algebra. $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ & I_{\mathcal{B}} \end{bmatrix}$ is a normal dual Banach $\mathcal{T}$-bimodule. If $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$ are w^*-continuous derivations, then $D_{\mathcal{AB}} : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ defined by*

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \longmapsto D_{\mathcal{A}}(a) + D_{\mathcal{B}}(b)$$

is a w^*-continuous derivation. Furthermore, $D_{\mathcal{AB}}$ is inner if and only if $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are inner.

Proof. Clearly, $D_{\mathcal{AB}}$ is a w^*-continuous derivation. Assume that $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are inner. Therefore there exist $x \in I_{\mathcal{A}}$ and $y \in I_{\mathcal{B}}$ such that $D_{\mathcal{A}}(a) = a \cdot x - x \cdot a$ and $D_{\mathcal{B}}(b) = b \cdot y - y \cdot b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. These lead to

$$\begin{aligned} D(\begin{bmatrix} a & m \\ & b \end{bmatrix}) &= D_{\mathcal{A}}(a) + D_{\mathcal{B}}(b) = (a \cdot x - x \cdot a) + (b \cdot y - y \cdot b) \\ &= (a \cdot x + m \cdot x + b \cdot x - x \cdot a - x \cdot m - x \cdot b) \\ &\quad +(a \cdot y + m \cdot y + b \cdot y - y \cdot a - y \cdot m - y \cdot b) \\ &= (a, m, b) \cdot (x + y) - (x + y) \cdot (a, m, b) \\ &= \begin{bmatrix} a & m \\ & b \end{bmatrix} \cdot (x + y) - (x + y) \cdot \begin{bmatrix} a & m \\ & b \end{bmatrix}. \end{aligned}$$

Hence, $D_{\mathcal{AB}} \in \mathcal{N}_{w^*}(\mathcal{T}, I_{\mathcal{T}})$. Converse is hold by the same method. □

Lemma 2 *Let $\mathcal{T}$ be a dual triangular Banach algebra, and $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & Y \\ & I_{\mathcal{B}} \end{bmatrix}$ If $D : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ is a w^*-continuous derivation, then there exist w^*-continuous derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$ and there is a w^*-continuous mapping $\theta : \mathcal{M} \longrightarrow Y$ such that*

1. *$\theta(a \cdot m) = a \cdot \theta(m) + D_{\mathcal{A}}(a) \cdot m$,*
2. *$\theta(m \cdot b) = \theta(m) \cdot b + m \cdot D_{\mathcal{B}}(b)$,*

for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$. Moreover, if D is inner then $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are inner.

Proof. Define $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$ by

$$D_{\mathcal{A}}(a) = e_{\mathcal{A}} \cdot D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}) \cdot e_{\mathcal{A}},$$

and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$ by

$$D_{\mathcal{B}}(b) = e_{\mathcal{B}} \cdot D(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) \cdot e_{\mathcal{B}},$$

for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Clearly, $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are w^*-continuous derivations. Consider the mapping $\theta : \mathcal{M} \longrightarrow Y$ via

$$\theta(m) = e_{\mathcal{A}} \cdot D(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) \cdot e_{\mathcal{B}},$$

for every $m \in \mathcal{M}$. By easy calculation one can show that θ satisfies on stated conditions. □

Theorem 3 *Let $\mathcal{T}$ be a dual triangular Banach algebra and $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ & I_{\mathcal{B}} \end{bmatrix}$ is a w^*-closed two sided ideal in $\mathcal{T}$. Then*

$$\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) \simeq \mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) \oplus \mathcal{H}^1_{w^*}(\mathcal{B}, I_{\mathcal{B}}). \tag{2}$$

Proof. It is obvious by ([18] theorem 2.7) □

Corollary 1 *Let $\mathcal{T}$ be a dual triangular Banach algebra. Then by above Theorem the following result immediately holds.*

(i) $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{A}}) \simeq \mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}})$.
(ii) $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{B}}) \simeq \mathcal{H}^1_{w^*}(\mathcal{B}, I_{\mathcal{B}})$.

Example 1 Let $I_{\mathcal{T}} = \begin{bmatrix} \mathcal{A} & 0 \\ & \mathcal{B} \end{bmatrix}$ where $\mathcal{A}$ is a Von-Neumann algebra. It is known that $\mathcal{A}$ is ideally connes amenable[17]. Then by applying Corollary 1, $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{A}}) = 0$ for every $w^\star$-closed two sided ideal $I_{\mathcal{A}}$in $\mathcal{A}$ similarly if $\mathcal{B}$ is a Von-Neumann algebra then $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{B}}) = 0$ for every $w^\star$-closed two sided ideal $I_{\mathcal{B}}$in $\mathcal{B}$.

Lemma 3 *([18]) Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Then $\mathcal{A} \oplus_\infty \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_1 \mathcal{B}_*$ and following product and norm:*

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2), \ \|(a, b)\| = \max\{\|a\|_{\mathcal{A}}, \|b\|_{\mathcal{B}}\}.$$

Corollary 2 *([18]) Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras. Then $\mathcal{A} \oplus_\infty \mathcal{B}$ is Connes-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are Connes-amenable.*

Theorem 4 *([18]) Let $\mathcal{T}$ be a dual triangular Banach algebra. Then $\mathcal{T}$ is Connes-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are Connes-amenable and $\mathcal{M} = 0$.*

Now, this question arise that for triangular dual Banach algebra $\mathcal{T}$, when $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = \{0\}$? From now on, suppose that $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras and $\mathcal{M}$ is a dual and normal Banach left $\mathcal{A}$-module and is a dual and normal Banach right $\mathcal{B}$-module, and finally, $\mathcal{T}$ is a dual triangular Banach algebra defined as before.

We start with the following Lemma that its proof is straightforward (see Proposition 2.1 of [7] and Lemma 2).

Lemma 4 *Let $\mathcal{T}$ be a dual triangular Banach algebra. $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & Y \\ & I_{\mathcal{B}} \end{bmatrix}$ an ideal in $\mathcal{T}$ and $D : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ be a w^*-continuous derivation. Then there are w^*-continuous derivation $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$, $m_D \in Y$ and w^*-continuous linear mapping $\theta : \mathcal{M} \longrightarrow Y$ such that:*

(i) $D(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}) = \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}$.

(ii) $D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}) = \begin{bmatrix} D_{\mathcal{A}}(a) & a \cdot m_D \\ & 0 \end{bmatrix}$.

(iii) $D(\begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}) = \begin{bmatrix} 0 & -m_D \\ & 0 \end{bmatrix}$.

(iv) $D(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) = \begin{bmatrix} 0 & -m_D \cdot b \\ & D_{\mathcal{B}}(b) \end{bmatrix}$.

(v) $D(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) = \begin{bmatrix} 0 & \theta(m) \\ & 0 \end{bmatrix}$.

(vi) $\theta(a \cdot m) = a \cdot \theta(m) + D_{\mathcal{A}}(a) \cdot m$.

(vii) $\theta(m \cdot b) = \theta(m) \cdot b + m \cdot D_{\mathcal{B}}(b)$.

Conversely, if $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$ *and* $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$ *are* w^**-continuous derivations and* $\theta : \mathcal{M} \longrightarrow Y$ *is a linear* w^**-continuous map that satisfies in conditions (vi) and (vii), then* $D : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ *defined by*

$$D(\begin{bmatrix} a & m \\ & b \end{bmatrix}) = \begin{bmatrix} D_{\mathcal{A}}(a) & \theta(m) \\ & D_{\mathcal{B}}(b) \end{bmatrix},$$

is a w^**-continuous derivation.*

Definition 1 Let $\mathcal{A}$,$\mathcal{B}$,$\mathcal{M}$, $I_{\mathcal{A}}$, $I_{\mathcal{B}}$, Y be as before

(i) For any $a \in I_{\mathcal{A}}$ and $b \in I_{\mathcal{B}}$, we say the w^*-continuous linear mapping $\theta_{a,b} : \mathcal{M} \longrightarrow Y$ is a w^*-Rosenblum operator on $\mathcal{M}$ with coefficients in Y if $\theta_{a,b}(m) = a \cdot m - m \cdot b$, for every $m \in \mathcal{M}$.

(ii) We say the w^*-continuous linear mapping $\theta : \mathcal{M} \longrightarrow Y$ is a w^*-generalized Rosenblum operator if there are w^*-continuous derivations $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow I_{\mathcal{B}}$ such that

$$\theta(a \cdot m \cdot b) = D_{\mathcal{A}}(a) \cdot m \cdot b + a \cdot \theta(m) \cdot b + a \cdot m \cdot D_{\mathcal{B}}(b),$$

for every $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$.

(iii) We shall denote the centralizer of $\mathcal{A}$ in $I_{\mathcal{A}}$ as $Z_{\mathcal{A}}(I_{\mathcal{A}}) = \{x \in I_{\mathcal{A}} : x.a = a.x \quad \forall\, a \in \mathcal{A}\}$ and the centralizer of $\mathcal{B}$ in $I_{\mathcal{B}}$ as $Z_{\mathcal{B}}(I_{\mathcal{B}}) = \{z \in I_{\mathcal{B}} : z.b = b.z \quad \forall\, b \in \mathcal{B}\}$. We say $\theta_{a,b}$ is a w^*-central Rosenblum operator on $\mathcal{M}$ with coefficients in Y if $a \in Z_{\mathcal{A}}(I_{\mathcal{A}})$ and $b \in Z_{\mathcal{B}}(I_{\mathcal{B}})$. We denote the space of all w^*-central Rosenblum operators by $ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$.

(iv) We denote the space of all w^*-continuous left $\mathcal{A}$-module morphisms and w^*-continuous right $\mathcal{B}$-module morphisms on $\mathcal{M}$ by

$$Hom_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y) = \{\varphi : \mathcal{M} \longrightarrow Y; \varphi(a.m.b) = a.\varphi(m).b \quad \forall\, a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}$$

, if $\mathcal{A} = \mathcal{B}$, we write $Hom_{w^*}^{\mathcal{A}}(\mathcal{M}, Y)$.

Lemma 5 *[7, Lemma 2.6, 2.7] Let* $\mathcal{T}$ *be a dual triangular Banach algebra. Then*

(i) $ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y) \subseteq \mathrm{Hom}_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$.

(ii) Let $\varphi \in \mathrm{Hom}_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$. Then $D_\varphi : \mathcal{T} \longrightarrow I_\mathcal{T}$ defined by

$$D_\varphi\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & \varphi(m) \\ & 0 \end{bmatrix}$$

is a w^*-continuous derivation. Moreover, D_φ is an inner derivation if and only if there exist $x \in Z_\mathcal{A}(I_\mathcal{A})$, $y \in Z_\mathcal{B}(I_\mathcal{B})$ such that $\varphi = \tau_{x,y} \in ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$.

Proof. (i) Let $\tau_{x,y} \in ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$. Then

$$\begin{aligned} \tau_{x,z}(a,m,b) &= xamb - ambz \\ &= axmb - amzb \\ &= a(xm - mz)b \\ &= a\tau_{x,z}(m)b. \end{aligned}$$

(ii) The first statement follows immediately from Lemma 4 when $D_\mathcal{A} = D_\mathcal{B} = 0$.
Assume that $\varphi = \tau_{x,z}$ where $x \in Z_\mathcal{A}(I_\mathcal{A})$, $z \in Z_\mathcal{B}(I_\mathcal{B})$. Then

$$\begin{aligned} \delta_{\begin{bmatrix} x & 0 \\ & z \end{bmatrix}}\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) &= \begin{bmatrix} x & 0 \\ & z \end{bmatrix}\begin{bmatrix} a & m \\ & b \end{bmatrix} - \begin{bmatrix} a & m \\ & b \end{bmatrix}\begin{bmatrix} x & 0 \\ & z \end{bmatrix} \\ &= \begin{bmatrix} xa - ax & xm - mz \\ & zb - bz \end{bmatrix} \\ &= \begin{bmatrix} 0 & xm - mz \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \varphi(m) \\ & 0 \end{bmatrix} = \delta_\varphi\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big). \end{aligned}$$

Hence δ_φ is inner. Conversely, assume that δ_φ is inner. Then there exists $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in I_\mathcal{T}$ such that $\delta_\varphi = \delta_{\begin{bmatrix} x & y \\ & z \end{bmatrix}}$. However

$$\begin{aligned} \delta_{\begin{bmatrix} x & y \\ & z \end{bmatrix}}\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) &= \begin{bmatrix} x & y \\ & z \end{bmatrix}\begin{bmatrix} a & m \\ & b \end{bmatrix} - \begin{bmatrix} a & m \\ & b \end{bmatrix}\begin{bmatrix} x & y \\ & z \end{bmatrix} \\ &= \begin{bmatrix} xa - ax & xm + yb - ay - mz \\ & zb - bz \end{bmatrix} \end{aligned}$$

If $\delta_\varphi = \delta_{\begin{bmatrix} x & y \\ & z \end{bmatrix}}$, then $xa - ax = 0$ for all $a \in \mathcal{A}$ and $zb - bz = 0$ for all $b \in \mathcal{B}$. In particular, $x \in Z_\mathcal{A}(I_\mathcal{A})$, $z \in Z_\mathcal{B}(I_\mathcal{B})$. Moreover, we have $\varphi(m) = xm + yb - ay - mz$.

Since $\varphi \in Hom_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$, it follows that $yb - ay = 0$. Hence $\varphi(m) = xm - mz = \tau_{x,z}(m)$.
In particular, $\varphi \in ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$.

□

Proposition 1 *[7, Theorem 2.8] Let $\mathcal{T}$ be a dual triangular Banach algebra. If $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) = 0$ and $\mathcal{H}^1_{w^*}(\mathcal{B}, I_{\mathcal{B}}) = 0$, then*

$$\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) \cong \frac{\mathrm{Hom}_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)}{ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)}.$$

Proof. Let $\Phi : Hom_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y) \longrightarrow \mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}})$ defined by: $\Phi(\varphi) = \overline{\delta}_{\varphi}$ where $\overline{\delta}_{\varphi}$ represents equivalence class of δ_{φ} in $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}})$. Clearly Φ is linear. We first show that Φ is surjective. Let $D : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ be a w^*-continuous derivation. Let $D_{\mathcal{A}}$, $D_{\mathcal{B}}$, θ, m_D be as in Lemma 4. Since $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) = 0$ and $\mathcal{H}^1_{w^*}(\mathcal{B}, I_{\mathcal{B}}) = 0$ we can find $x \in I_{\mathcal{A}}$, $z \in I_{\mathcal{B}}$ such that $D_{\mathcal{A}} = \delta_x$ and $D_{\mathcal{B}} = \delta_z$. Define $D_0 : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ by

$$D_0(\begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \delta_x(a) & \mathcal{T}_{x,z}(m) + (a.m_D - m_D.b) \\ & \delta_z(b) \end{bmatrix},$$

Then D_0 is the inner derivation induced by $\mathcal{T} = \begin{bmatrix} x & -m_D \\ & z \end{bmatrix}$ and as such D_0 is clearly w^*-continuous. Further more if $D_1 = D - D_0$ then D_1 is a w^*-continuous derivation and due to Lemma 4

$$D_1(\begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \delta_x(a) & \theta(m) + (a.m_D - m_D.b) \\ & \delta_z(b) \end{bmatrix} - \begin{bmatrix} \delta_x(a) & \mathcal{T}_{x,z}(m) + (a.m_D - m_D.b) \\ & \delta_z(b) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \theta(m) - \mathcal{T}_{x,z}(m) \\ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{T}_1(m) \\ & 0 \end{bmatrix}$$

where $\mathcal{T}_1 = \theta - \mathcal{T}_{x,z}$.
It is easy to see that $\mathcal{T}_1 \in Hom_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$. Finally $\overline{D} = \overline{D_1} = \Phi(\mathcal{T}_1)$, and so Φ is surjective. We have shown that

$$\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) \cong \frac{Hom_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)}{Ker\Phi}.$$

However $\varphi \in Ker\Phi$ if and only if δ_{φ} is inner. By lemma5 $Ker\Phi = ZR_{w^*}^{\mathcal{A},\mathcal{B}}(\mathcal{M}, Y)$

□

Let $\mathcal{A}$ be a unital dual Banach algebra and consider $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix}$ and $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & I_{\mathcal{A}} \\ & I_{\mathcal{A}} \end{bmatrix}$ then in light of Lemma 4, if $D_{\mathcal{A}} : \mathcal{A} \longrightarrow I_{\mathcal{A}}$ is a w^*-continuous derivation then $D : \mathcal{T} \longrightarrow I_{\mathcal{T}}$ defined by

$$D(\begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} D_{\mathcal{A}}(a) & D_{\mathcal{A}}(m) \\ & D_{\mathcal{A}}(b) \end{bmatrix},$$

is a w^*-continuous derivation. Moreover, D is inner if and only if $D_{\mathcal{A}}$ is inner. It follows immediately that there exists a linear isomorphism from $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}})$ onto a subspace $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}})$ [7, Corllary 3.2]. Hence if $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = 0$ then $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) = 0$. Therefore we can write the following result that its proof is similar to proof of Proposition 3.3 of [7] but replace identity map $id : \mathcal{A} \longrightarrow \mathcal{A}$ by natural projection map from $\mathcal{A}$ onto $I_{\mathcal{A}}$ which is clearly in $Hom^{\mathcal{A}}_{w^*}(\mathcal{A}, I_{\mathcal{A}})$.

Proposition 2 *Let $\mathcal{A}$ be a dual Banach algebra (non-unital) and* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix}$ *be a dual triangular Banach algebra and* $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & I_{\mathcal{A}} \\ & I_{\mathcal{A}} \end{bmatrix}$. *If $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = 0$ then $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) = 0$ and $\mathcal{A}$ is unital.*

Proposition 3 *Let $\mathcal{A}$ be a unital dual Banach algebra and $\mathcal{T}$, $I_{\mathcal{T}}$ be the above defined dual triangular Banach algebras, then $\mathcal{H}^1_{w^*}(\mathcal{A}, I_{\mathcal{A}}) = 0$ if and only if $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = 0$.*

Proof. Lemma 4.3 of [8] leads to $Hom^{\mathcal{A}}_{w^*}(\mathcal{A}, I_{\mathcal{A}}) \simeq ZR^{\mathcal{A}}_{w^*}(\mathcal{A}, I_{\mathcal{A}})$ and Proposition 1 implies that $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = 0$. Proposition 2 implies that the converse assertion also holds true. □

Example 2 Let $\mathcal{A}$ be a Von-Neumann algebra or $\mathcal{A} = B(G)$, Fourier Stieltjes algebra of G where G is a locally compact amenable group, hence $\mathcal{A}$ is a unital dual Banach algebra, let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{A} \\ & \mathcal{A} \end{bmatrix}$ and $I_{\mathcal{T}} = \begin{bmatrix} I_{\mathcal{A}} & I_{\mathcal{A}} \\ & I_{\mathcal{A}} \end{bmatrix}$. Then in light of ideal Connes-amenability $\mathcal{A}$ [17] and Proposition 3, we have $\mathcal{H}^1_{w^*}(\mathcal{T}, I_{\mathcal{T}}) = 0$.

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