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Fixed Points of $(\psi, \varphi)_\Omega$-Contractive Mappings in Ordered P-Metric Spaces

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Abstract In this paper, we introduce the notion of an extended metric space (p-metric space) as a new generalization of the concept of b-metric space. Also, we present the concept of $(\psi, \varphi)_\Omega$-contractive mappings and we establish some fixed point results for this class of mappings in ordered complete p-metric spaces. Our results generalize several well-known comparable results in the literature. Finally, examples support our results.

Keywords Fixed point · Complete metric space · Ordered b-metric space

Mathematics Subject Classification (2010) 47H10 · 54H25

1 Introduction

The Banach contraction principle [5] is a very powerful tool for solving problems in nonlinear analysis. Some authors generalized this interesting theorem in different ways (see, *e.g.*, [1, 2, 7, 8, 10, 11, 14, 18]).

Khan *et al.* [17] introduced the concept of an altering distance function as follows.

Definition 1 [17] The function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function, if the following properties hold:

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1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, *e.g.*, [1,12,15,17,19,21,23,24]).

The concept of a b-metric space was introduced by Czerwik in [9]. After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in b-metric spaces have been obtained (see, [1,3,4,6,13,16,20,22,26]).

Definition 2 [9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric iff for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

A b-metric is a metric, when $s = 1$.

Motivated with [9], the following definitions and results will be needed in the sequel.

Definition 3 Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to R^+$ is a p-metric iff there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $x \leq \Omega(x)$ such that for all $x, y, z \in X$, the following conditions hold:

(p_1) $\widetilde{d}(x, y) = 0$ iff $x = y$,
(p_2) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$,
(p_3) $\widetilde{d}(x, z) \leq \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z))$.

In this case, the pair $(X, \widetilde{d})$ is called a p-metric space, or, an extended b-metric space.

It should be noted that, the class of p-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is a p-metric, when $\Omega(x) = sx$ while a metric is a p-metric, when $\Omega(x) = x$.

Here, we present an example to show that in general, a p-metric need not necessarily to be a b-metric.

Example 1 Let $(X, \widetilde{d})$ be a metric space and $\rho(x, y) = \sinh \widetilde{d}(x, y)$. We show that ρ is a p-metric with $\Omega(t) = \sinh(t)$ for all $t \geq 0$.

Obviously, conditions (p_1) and (p_2) of Definition 3 are satisfied.

For each $x, y, z \in X$,

$$\begin{array}{rl} & \rho(x,y) \\ & = \sinh(\widetilde{d}(x,y)) \leq \sinh(\widetilde{d}(x,z)+\widetilde{d}(z,y)) \\ & \leq \sinh[\sinh(\widetilde{d}(x,z))+\sinh(\widetilde{d}(z,y))] \quad rl \\ & = \sinh(\rho(x,z)+\rho(z,y)) \\ & = \Omega(\rho(x,z)+\rho(z,y)). \end{array} \tag{1}$$

So, condition (p_3) of Definition 3 is also satisfied and ρ is a p-metric. Note that, $sinh|x-y|$ is not a metric on $\mathcal{R}$, as we know that

$$\sinh 5 = 74.2032105778 \geq 3.62686040785 + 10.0178749274 = \sinh 2 + \sinh 3.$$

Obviously, $sinh|x-y|$ is not also a b-metric for any $s \geq 1$.

Example 2 Let $(X, \widetilde{d})$ be a metric space and $\rho(x,y) = e^{\widetilde{d}(x,y)} - 1$. We show that ρ is a p-metric with $\Omega(t) = e^t - 1$.

Obviously, conditions (p_1) and (p_2) of Definition 3 are satisfied.

On the other hand, for each $x, y, z \in X$,

$$\begin{array}{l} \rho(x,y) \\ = e^{\widetilde{d}(x,y)} - 1 \leq e^{\widetilde{d}(x,z)+\widetilde{d}(z,y)} - 1 \\ \leq e^{e^{\widetilde{d}(x,z)}-1+e^{\widetilde{d}(z,y)}-1} - 1 \\ = e^{(\rho(x,z)+\rho(z,y))} - 1 \\ = \Omega(\rho(x,z)+\rho(z,y)). \end{array} \tag{2}$$

So, condition (p_3) of Definition 3 is also satisfied and ρ is a p-metric.

In general, we have the following proposition.

Proposition 1 *Let $(X, \widetilde{d})$ be a metric space with coefficient $s \geq 1$ and let $\rho(x,y) = \xi(\widetilde{d}(x,y))$ where $\xi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $x \leq \xi(x)$ and $0 = \xi(0)$. We show that ρ is a p-metric with $\Omega(t) = \xi(t)$.*

For each $x, y, z \in X$,

$$\begin{array}{l} \rho(x,y) \\ = \xi(\widetilde{d}(x,y)) \leq \xi(\widetilde{d}(x,z)+\widetilde{d}(z,y)) \\ \leq \xi(\xi(\widetilde{d}(x,z))+\xi(\widetilde{d}(z,y)) \\ = \Omega(\rho(x,z)+\rho(z,y)). \end{array} \tag{3}$$

So, ρ is a p-metric.

The above proposition constructs the following example:

Example 3 Let $(X, \widetilde{d})$ be a metric space and let $\rho(x,y) = e^{\widetilde{d}(x,y)} \sec^{-1}(e^{\widetilde{d}(x,y)})$. Then ρ is a p-metric with $\Omega(t) = e^t \sec^{-1}(e^t)$.

Definition 4 Let $(X, \widetilde{d})$ be a p-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) p-convergent if and only if there exists $x \in X$ such that $\widetilde{d}(x_n, x) \to 0$, as $n \to +\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(b) p-Cauchy if and only if $\widetilde{d}(x_n, x_m) \to 0$ as $n, m \to +\infty$.

(c) The p-metric space $(X, \widetilde{d})$ is p-complete if every p-Cauchy sequence in X p-converges.

Proposition 2 *In a p-metric space $(X, \widetilde{d})$, as $\Omega(0) = 0$,*

p_1. *A p-convergent sequence has a unique limit.*

p_2. *Each p-convergent sequence is p-Cauchy.*

p_3. *In general, a p-metric is not continuous.*

We will need the following simple lemma about the p-convergent sequences.

Lemma 1 *Let $(X, \widetilde{d})$ be a p-metric space with a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$, and suppose that $\{x_n\}$ and $\{y_n\}$ p-converge to x, y, respectively. Then, we have*

$$(\Omega^2)^{-1}(\widetilde{d}(x, y)) \leq \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n) \leq \Omega^2(\widetilde{d}(x, y)).$$

In particular, if $x = y$, then, $\lim_{n\longrightarrow\infty}\widetilde{d}(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\Omega^{-1}(\widetilde{d}(x, z)) \leq \liminf_{n\longrightarrow\infty} \widetilde{d}(x_n, z) \leq \limsup_{n\longrightarrow\infty} \widetilde{d}(x_n, z) \leq \Omega(\widetilde{d}(x, z)).$$

Proof. (a) Using the p-triangular inequality, it is easy to see that

$$\begin{aligned}\widetilde{d}(x, y) &\leq \Omega(\widetilde{d}(x, x_n) + \widetilde{d}(x_n, y)) \\ &\leq \Omega(\widetilde{d}(x, x_n) + \Omega(\widetilde{d}(x_n, y_n) + \widetilde{d}(y_n, y)))\end{aligned} \tag{4}$$

and

$$\widetilde{d}(x_n, y_n) \leq \Omega(\widetilde{d}(x_n, x) + \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, y_n))).$$

Taking the lower limit as $n \to \infty$ in the first inequality one has

$$\widetilde{d}(x, y) \leq \Omega(\Omega(\liminf_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n)))$$

and taking the upper limit as $n \to \infty$ in the second inequality we have

$$\limsup_{n\longrightarrow\infty} \widetilde{d}(x_n, y_n) \leq \Omega(\Omega(\widetilde{d}(x, y))).$$

(b) Using the p-triangular inequality, it is easy to see that

$$\widetilde{d}(x, z) \leq \Omega(\widetilde{d}(x, x_n) + \widetilde{d}(x_n, z))$$

and

$$\widetilde{d}(x_n, z) \leq \Omega(\widetilde{d}(x_n, x) + \widetilde{d}(x, z)).$$

Taking the lower limit as $n \to \infty$ in the first inequality one has

$$\widetilde{d}(x,z) \leq \Omega(\liminf_{n\longrightarrow\infty} \widetilde{d}(x_n,z))$$

and taking the upper limit as $n \to \infty$ in the second inequality we have

$$\limsup_{n\longrightarrow\infty} \widetilde{d}(x_n,z) \leq \Omega(\widetilde{d}(x,z)).$$

□

In this paper, we introduce the notion of generalized $(\psi,\varphi)_\Omega$-contractive mapping and we establish some results in complete ordered p-metric spaces, where ψ and φ are altering distance functions. Our results generalize several comparable results in the literature.

2 Main results

In this section, we define the notion of $(\psi,\varphi)_\Omega$-contractive mapping and prove our new results.

Let $(X,\preceq,d)$ be an ordered p-metric space and let $f : X \to X$ be a mapping. Set

$$M(x,y) = \max\left\{\widetilde{d}(x,y), \widetilde{d}(x,fx), \widetilde{d}(y,fy), \widetilde{d}(y,fx)\right\}. \tag{5}$$

Definition 5 Let $(X,\preceq,d)$ be an ordered p-metric space. We say that a mapping $f : X \to X$ is an ordered $(\psi,\varphi)_\Omega$-contractive mapping if there exist two altering distance functions ψ and φ and strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $x \leq \Omega(x)$, for all nonnegative real number x such that

$$\psi(\Omega(\widetilde{d}(fx,fy))) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{6}$$

for all comparable elements $x, y \in X$.

Now, let us to prove our first result.

Theorem 1 *Let $(X,\preceq,d)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be an ordered non-decreasing continuous ordered $(\psi,\varphi)_\Omega$-contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

Proof: Let $x_0 \in X$ be arbitrary. Define a sequence (x_n) in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$. Inductively, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_n = x_{n+1}$, for some $n \in \mathcal{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathcal{N}$. By (6), we have

$$\begin{aligned} \psi(\widetilde{d}(x_n, x_{n+1})) &\leq \psi(\Omega(\widetilde{d}(x_n, x_{n+1}))) \\ &= \psi(\Omega(\widetilde{d}(fx_{n-1}, fx_n))) \\ &\leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)), \end{aligned} \tag{7}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_{n-1}, fx_{n-1}), \widetilde{d}(x_n, fx_n), \widetilde{d}(x_n, fx_{n-1})\right\} \\ &= \max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\} \\ &= \max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\}. \end{aligned} \tag{8}$$

From (7) and (8) and the properties of ψ and φ, we get

$$\begin{aligned} \psi(\widetilde{d}(x_n, x_{n+1})) &\leq \psi\left(\max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\}\right) \\ &\quad -\varphi\left(\max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\}\right) \\ &< \psi\left(\max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\}\right). \end{aligned} \tag{9}$$

If

$$\max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\} = \widetilde{d}(x_n, x_{n+1}),$$

then by (9) we have

$$\begin{aligned} \psi(\widetilde{d}(x_n, x_{n+1})) &\leq \psi(\widetilde{d}(x_n, x_{n+1})) - \varphi(\widetilde{d}(x_n, x_{n+1})) \\ &< \psi(\widetilde{d}(x_n, x_{n+1})), \end{aligned} \tag{10}$$

which gives a contradiction. Thus,

$$\max\left\{\widetilde{d}(x_{n-1}, x_n), \widetilde{d}(x_n, x_{n+1})\right\} = \widetilde{d}(x_{n-1}, x_n).$$

Therefore (9) becomes

$$\psi(\widetilde{d}(x_n, x_{n+1})) \leq \psi(\widetilde{d}(x_n, x_{n-1})) - \varphi(\widetilde{d}(x_{n-1}, x_n)) < \psi(\widetilde{d}(x_n, x_{n-1})). \tag{11}$$

Since ψ is a non-decreasing mapping, $\{\widetilde{d}(x_n, x_{n+1}) : n \in \mathcal{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \widetilde{d}(x_n, x_{n+1}) = r. \tag{12}$$

Letting $n \to \infty$ in (11), we get

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r). \tag{13}$$

Therefore, $\varphi(r)=0$, and hence $r=0$. Thus, we have

$$\lim_{n\to\infty}\widetilde{d}(x_n,x_{n+1})=0. \tag{14}$$

Next, we show that $\{x_n\}$ is a p-Cauchy sequence in X. By contradiction, there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i,\ \ \widetilde{d}(x_{m_i},x_{n_i})\geq\varepsilon. \tag{15}$$

This means that

$$\widetilde{d}(x_{m_i},x_{n_i-1})<\varepsilon. \tag{16}$$

From (15) and using the p-triangular inequality, we get

$$\begin{aligned}\varepsilon&\leq\widetilde{d}(x_{m_i},x_{n_i})\\&\leq\Omega(\widetilde{d}(x_{m_i},x_{m_i-1})+\widetilde{d}(x_{m_i-1},x_{n_i}))\\&\leq\Omega(\widetilde{d}(x_{m_i},x_{m_i-1})+\Omega(\widetilde{d}(x_{m_i-1},x_{n_i-1})+\widetilde{d}(x_{n_i-1},x_{n_i}))).\end{aligned} \tag{17}$$

Using (17) and taking the upper limit as $i\to\infty$, we get

$$(\Omega^2)^{-1}(\varepsilon)\leq\liminf_{i\longrightarrow\infty}\widetilde{d}(x_{m_i-1},x_{n_i-1}). \tag{18}$$

On the other hand, we have

$$\widetilde{d}(x_{m_i-1},x_{n_i-1})\leq\Omega(\widetilde{d}(x_{m_i-1},x_{m_i})+\widetilde{d}(x_{m_i},x_{n_i-1})). \tag{19}$$

Using (14), (16) and taking the upper limit as $i\to\infty$, we get

$$\limsup_{i\longrightarrow\infty}\widetilde{d}(x_{m_i-1},x_{n_i-1})\leq\Omega(\varepsilon). \tag{20}$$

On the other hand, we have

$$\widetilde{d}(x_{m_i},x_{n_i})\leq\Omega(\widetilde{d}(x_{m_i},x_{n_i-1})+\widetilde{d}(x_{n_i-1},x_{n_i})). \tag{21}$$

Using (14), (15) and taking the upper limit as $i\to\infty$, we get

$$\limsup_{i\longrightarrow\infty}\widetilde{d}(x_{m_i},x_{n_i-1})\geq\Omega^{-1}(\varepsilon). \tag{22}$$

From (6), we have

$$\begin{aligned}\psi(\Omega(\widetilde{d}(x_{m_i},x_{n_i})))&=\psi(\Omega(\widetilde{d}(fx_{m_i-1},fx_{n_i-1})))\\&\leq\psi(M(x_{m_i-1},x_{n_i-1}))-\varphi(M(x_{m_i-1},x_{n_i-1})),\end{aligned} \tag{23}$$

where

$$\begin{aligned}M(x_{m_i-1},x_{n_i-1})&=\max\left\{\widetilde{d}(x_{m_i-1},x_{n_i-1}),\widetilde{d}(x_{m_i-1},fx_{m_i-1}),\widetilde{d}(x_{n_i-1},fx_{n_i-1}),\widetilde{d}(x_{n_i-1},fx_{m_i-1})\right\}\\&=\max\left\{\widetilde{d}(x_{m_i-1},x_{n_i-1}),\widetilde{d}(x_{m_i-1},x_{m_i}),\widetilde{d}(x_{n_i-1},x_{n_i}),\widetilde{d}(x_{n_i-1},x_{m_i})\right\}.\end{aligned} \tag{24}$$

Taking the upper limit as $i \to \infty$ in (24) and using (14), we get

$$\limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) = \max\{\limsup_{i\longrightarrow\infty} \widetilde{d}(x_{m_i-1}, x_{n_i-1}), 0, 0, \limsup_{i\longrightarrow\infty} \widetilde{d}(x_{m_i}, x_{n_i-1})\} \le \Omega(\varepsilon). \tag{25}$$

So, we have

$$\limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) \le \Omega(\varepsilon), \tag{26}$$

Similarly, we obtain that

$$(\Omega^2)^{-1}(\varepsilon) \le \liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}). \tag{27}$$

Now, taking the upper limit as $i \to \infty$ in (23) and using (26) and (27), we have

$$\begin{aligned}\psi(\Omega(\varepsilon)) &\le \psi(\Omega(\limsup_{i\longrightarrow\infty} \widetilde{d}(x_{m_i}, x_{n_i}))) \\ &\le \psi(\limsup_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})) - \liminf_{i\longrightarrow\infty} \varphi(M(x_{m_i-1}, x_{n_i-1})) \\ &\le \psi(\Omega(\varepsilon)) - \varphi(\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})),\end{aligned} \tag{28}$$

which further implies that

$$\varphi(\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1})) = 0,$$

so $\liminf_{i\longrightarrow\infty} M(x_{m_i-1}, x_{n_i-1}) = 0$, a contradiction to (27). Thus, $\{x_{n+1} = fx_n\}$ is a p-Cauchy sequence in X. As X is a p-complete space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, and

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = u.$$

Now, as f is continuous, using the p-triangular inequality, we get

$$\widetilde{d}(u, fu) \le \Omega(\widetilde{d}(u, fx_n) + \widetilde{d}(fx_n, fu)).$$

Letting $n \to \infty$, we get

$$\widetilde{d}(u, fu) \le \Omega(\lim_{n\to\infty} \widetilde{d}(u, fx_n) + \lim_{n\to\infty} \widetilde{d}(fx_n, fu)) = 0.$$

So, we have $fu = u$. Thus, u is a fixed point of f.

Note that the continuity of f in Theorem 1 is not necessary and can be dropped.

Recall that, an ordered p-metric space $(X, \preceq, p)$ is said to have sequential limit comparison property (s.l.c.p) if for every nondecreasing sequence $\{x_n\}$ in X, converging to some $x \in X$, $x_n \preceq x$ holds for all $n \in \mathcal{N}$.

Theorem 2 *Under the same hypotheses of Theorem 1, without the continuity assumption of f, assume that $(X, \preceq, p)$ enjoys the s.l.c.p.. Then f has a fixed point in X.*

Proof. Following the proof of Theorem 1, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u$, for some $u \in X$. Using the assumption *s.l.c.p.* on X, we have $x_n \preceq u$, for all $n \in \mathcal{N}$. Now, we show that $fu = u$. By (6), we have

$$\begin{aligned}\psi(\Omega(\widetilde{d}(x_{n+1}, fu))) &= \psi(\Omega(\widetilde{d}(fx_n, fu)) \\ &\leq \psi(M(x_n, u)) - \varphi(M(x_n, u)),\end{aligned} \tag{29}$$

where

$$\begin{aligned}M(x_n, u) &= \max\left\{\widetilde{d}(x_n, u), \widetilde{d}(x_n, fx_n), \widetilde{d}(u, fu), \widetilde{d}(fx_n, u)\right\} \\ &= \max\left\{\widetilde{d}(x_n, u), \widetilde{d}(x_n, x_{n+1}), \widetilde{d}(u, fu), \widetilde{d}(x_{n+1}, u)\right\}.\end{aligned} \tag{30}$$

Letting $n \to \infty$ in (30) and using Lemma 1, we get

$$\limsup_{i\longrightarrow\infty} M(x_n, u) = \widetilde{d}(u, fu). \tag{31}$$

Similarly, we can obtain

$$\liminf_{i\longrightarrow\infty} M(x_n, u) = \widetilde{d}(u, fu). \tag{32}$$

Again, taking the upper limit as $n \to \infty$ in (29) and using Lemma 1 and (31) we get

$$\begin{aligned}\psi(\widetilde{d}(u, fu) &= \psi(\Omega(\Omega^{-1}(\widetilde{d}(u, fu))) \leq \psi(\Omega(\limsup_{i\longrightarrow\infty} \widetilde{d}(x_{n+1}, fu))) \\ &\leq \psi(\limsup_{i\longrightarrow\infty} M(x_n, u)) - \liminf_{i\longrightarrow\infty} \varphi(M(x_n, u)) \\ &\leq \psi(\widetilde{d}(u, fu)) - \varphi(\liminf_{i\longrightarrow\infty} M(x_n, u)).\end{aligned} \tag{33}$$

Therefore, $\varphi(\liminf_{n\longrightarrow\infty} M(x_n, u)) \leq 0$, equivalently, $\liminf_{n\longrightarrow\infty} M(x_n, u) = 0$. Thus, from (32) we get $u = fu$ and hence u is a fixed point of f.

Corollary 1 *Let $(X, \preceq, d)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be an ordered non-decreasing mapping. Suppose that there exist $k \in [0, 1)$ such that*

$$\Omega(\widetilde{d}(fx, fy)) \leq k \max\left\{\widetilde{d}(x, y), \widetilde{d}(x, fx), \widetilde{d}(y, fy), \widetilde{d}(y, fx)\right\},$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point provided that f is continuous, or, $(X, \preceq, p)$ enjoys the s.l.c.p..

Proof. Follows from Theorems (1) and (2) and by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$.

Corollary 2 *Let $(X, \preceq, d)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be an ordered non-decreasing mapping. Suppose that there exist $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + \gamma + \delta \in [0, 1)$ such that*

$$\Omega(\widetilde{d}(fx, fy)) \leq \alpha\widetilde{d}(x, y) + \beta\widetilde{d}(x, fx) + \gamma\widetilde{d}(y, fy) + \delta\widetilde{d}(y, fx),$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point provided that f is continuous, or, $(X, \preceq, p)$ enjoys the s.l.c.p..

The following corollary is an extension of Banach contraction principle in an extended b-metric space.

Corollary 3 *Let $(X, \preceq, d)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be an ordered non-decreasing mapping. Suppose that there exist $\alpha \in [0, 1)$ such that*

$$\sinh(\widetilde{d}(fx, fy)) \leq \alpha\widetilde{d}(x, y),$$

for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point provided that f is continuous, or, $(X, \preceq, p)$ enjoys the s.l.c.p..

Now, in order to support the usability of our results, we present the following examples.

Example 4 Let $X = [0, 100]$ be equipped with the p-metric $\widetilde{d}(x, y) = e^{|x-y|^2} - 1$ for all $x, y \in X$, where $\Omega(x) = e^{2x} - 1$.

Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$, the function $f : X \to X$ by

$$fx = \ln(1 + \frac{x}{10})$$

and the altering distance functions $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = \ln(1 + \frac{1}{2}\ln(1 + t))$ and $\varphi(t) = \frac{t}{1000}$. Then, we have the following:

1. $(X, \preceq, d)$ is a partially ordered p-complete p-metric space.
2. f is an ordered increasing mapping.
3. f is continuous.
4. f is an ordered $(\psi, \varphi)_\Omega$-contractive mapping, that is,

$$\psi(\Omega(\widetilde{d}(fx, fy))) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\left\{\widetilde{d}(x, y), \widetilde{d}(x, fx), \widetilde{d}(y, fy), \widetilde{d}(y, fx)\right\}.$$

Proof. The proof of (1), (2) and (3) is clear.

To prove (4), let $x, y \in X$ with $x \preceq y$. So, $y \leq x$. Thus, using the mean value theorem for function $\ln(1+\frac{t}{10})$, we have

$$\begin{aligned}
\psi\Big(\Omega(\widetilde{d}(fx,fy))\Big) &= \ln\Big(1+\tfrac{1}{2}\ln(1+\Omega(\widetilde{d}(fx,fy)))\Big) \\
&= \ln\left(1+\tfrac{1}{2}\left[\ln\left(1+e^{2e^{\left[\ln\left(1+\frac{x}{10}\right)-\ln\left(1+\frac{y}{10}\right)\right]^2}-2}-1\right)\right]\right) \\
&= [\ln(1+\tfrac{x}{10})-\ln(1+\tfrac{y}{10})]^2 \\
&\leq |\tfrac{x}{10}-\tfrac{y}{10}|^2 \\
&\leq \tfrac{1}{100}|x-y|^2 \\
&\leq \tfrac{1}{100}\left[e^{\left|x-y\right|^2}-1\right] \\
&\leq \tfrac{1}{100}[M(x,y)] \\
&\leq \ln(1+\tfrac{1}{2}\ln(1+M(x,y)))-\tfrac{1}{1000}[M(x,y)] \\
&= \psi(M(x,y))-\varphi(M(x,y)).
\end{aligned} \tag{34}$$

So, we conclude that f is a $(\psi,\varphi)_\Omega$-contractive mapping. Thus, all the hypotheses of Theorem 1 are satisfied and hence f has a fixed point. Indeed, 0 is the unique fixed point of f.

Remark 1 A subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable. Note that in Theorems 1 and 2, f has a unique fixed point provided that the fixed points of f are comparable.

Example 5 Let $X = \{0,1,2,3\}$ be equipped with the following partial order $\preceq$:

$$\preceq := \{(0,0),(1,0),(1,1),(1,2),(2,2),(3,1),(3,2),(3,3)\}.$$

Define the metric $\widetilde{d} : X \times X \to \mathcal{R}^+$ by

$$\widetilde{d}(x,y) = \begin{cases} 0, & x = y, \\ x+y, & x \neq y \end{cases} \tag{35}$$

and let $\rho(x,y) = \sinh \widetilde{d}(x,y)$. It is easy to see that (X,ρ) is a p-complete p-metric space.

Define the self-map f by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We see that f is an ordered increasing mapping and $(X,\preceq,p)$ enjoys the *s.l.c.p.*.

Define $\psi,\varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \sqrt{t}$ and $\varphi(t) = \frac{1}{1+t^2}$. One can easily check that f is a $(\psi,\varphi)_\Omega$-contractive mapping. Indeed, we have some cases as follows:

1. $(x, y) = (3, 1)$. Then,

$$\begin{aligned}\psi(\Omega(\rho(fx, fy))) &= \sqrt{\sinh(f3 + f1)} \\ &= \sqrt{\sinh(1 + 0)} \\ &= 1.08406696917 \\ &\leq 5.22397522938 - 0.00134095068 \\ &= \sqrt{M(x, y)} - \tfrac{1}{1+(M(x,y))^2} \\ &= \psi(M(x, y)) - \varphi(M(x, y)).\end{aligned} \tag{36}$$

2. $(x, y) = (3, 2)$. Then,

$$\begin{aligned}\psi(\Omega(\widetilde{d}(fx, fy))) &= \sqrt{\sinh(f3 + f2)} \\ &= \sqrt{\sinh(1 + 1)} \\ &= 1.90443178083 \\ &\leq 8.6141285443 - 0.00018158323 \\ &= \sqrt{M(x, y)} - \tfrac{1}{1+(M(x,y))^2} \\ &= \psi(M(x, y)) - \varphi(M(x, y)).\end{aligned} \tag{37}$$

Thus, all the conditions of Theorem 2 are satisfied and hence f has a fixed point. Indeed, 0 is the fixed point of f.

3 Existence theorem for a solution of an integral equation

Consider the integral equation

$$x(t) = p(t) + \int_0^T \lambda(t, r) f(r, x(r)) dr, \qquad t \in [0, T] \tag{38}$$

where $0 < T$. The purpose of this section is to give an existence theorem for a solution of 38 that belongs to $X = C(I, \mathcal{R})$ (the set of continuous real functions defined on $I = [0, T]$), via the obtained result in Theorem 2. Obviously, this space with the p-metric given by

$$\rho(x, y) = e^{\left(\max_{t \in I} \left|x(t) - y(t)\right|\right)} - 1$$

for all $x, y \in X$ is a p-complete p-metric space with $\Omega(t) = e^t - 1$.

We endow X with the partial order $\preceq$ given by

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),$$

for all $t \in I$. $(X, \preceq, \rho)$ is regular [25]. We will consider 38 under the following assumptions:

(i) $f, p : [0, T] \times \mathcal{R} \to \mathcal{R}$ are continuous.
(ii) $\lambda : [0, T] \times \mathcal{R} \to [0, \infty)$ is continuous.

(iii) There exists $k \in (0,1)$ such that for all x, y with $x \preceq y$

$$0 \leq e^{\left|\int_0^T \lambda(t,r)[f(r,x(r))-f(r,y(r))]dr\right|} - 1 \leq k[e^{(y(t)-x(t))} - 1],$$

and $\ln(1+t) - 2kt \geq 0$ for all $t \in I$.

(iv) $\max\limits_{t\in I} \int_0^T |\lambda(t,r)|dr \leq 1$.

(v) There exists continuous function $\alpha : [0,T] \to \mathcal{R}$ such that

$$\alpha(t) \leq p(t) + \int_0^T \lambda(t,r)f(r,\alpha(r))dr.$$

Theorem 3 *Under assumptions (i)-(v), 38 has a solution in* X*, where* $X = C([0,T],\mathcal{R})$.

Proof. We define $F : X \to X$ by

$$F(x(t)) = p(t) + \int_0^T \lambda(t,r)f(r,x(r))dr.$$

The mapping F is ordered increasing since, for $x \preceq y$

$$f(t,x) \leq f(t,y),$$

and from $\lambda(t,r) > 0$, we have

$$F(x(t)) = p(t) + \int_0^T \lambda(t,r)f(r,x(r))dr \leq p(t) + \int_0^T \lambda(t,r)f(r,y(r))dr = F(y(t)).$$

Now, we have

$$\begin{aligned}
\psi\Big(\Omega(\rho(Fx(t),Fy(t)))\Big) &= \ln\Big(\Omega(e^{|Fx(t)-Fy(t)|} - 1) + 1\Big) \\
&= \ln\Big(e^{e^{\left|Fx(t)-Fy(t)\right|}-1} - 1 + 1\Big) \\
&\leq e^{\left|\int_0^T \lambda(t,r)[f(r,x(r))-f(r,y(r))]dr\right|} - 1 \\
&\leq k[e^{(y(t)-x(t))} - 1] \\
&\leq k\rho(x,y) \\
&\leq kM(x,y) \\
&\leq \ln(M(x,y)+1) - kM(x,y) \\
&= \psi(M(x,y)) - \varphi(M(x,y)).
\end{aligned} \tag{39}$$

where

$$M(x,y) = \max\Big\{\rho(x,y), \rho(x,Fx), \rho(y,Fy), \rho(y,Fx)\Big\}.$$

Let α be the function appearing in assumption (v). Then we get

$$\alpha \preceq F(\alpha).$$

Thus, from Theorem 2 by $\psi(t) = \ln(1+t)$ and $\varphi(t) = kt$ we deduce the existence of an $x \in X$ such that $x = F(x)$. □

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