

On Bounded Weak Approximate Identities and a New Version of Them

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Abstract In this paper, we give a short survey of results and problems concerning the notion of bounded weak approximate identities in Banach algebras. Also, we introduce a new version of approximate identities and give one illuminating example to show the difference.

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1 Introduction

In [4], Jones and Lahr introduced a new notion of an approximate identity for a commutative Banach algebra A called "bounded weak approximate identity" and gave an example of a semi-simple and commutative semigroup algebra with a bounded weak approximate identity which has no approximate identity, bounded or unbounded. Several authors have studied the notion of bounded weak approximate identity; see [1, 2, 5, 8, 10, 11, 13, 14, 20–22].

In this short note we provide a number of results and applications of bounded weak approximate identities in the recent decades and finally we raise some open problems in this area which seems to be good questions for

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launching research about various types and properties of approximate identities.

2 Main definition and example

Let A be a Banach algebra, $\Delta(A)$ be the character space of A , i.e., the space of all non-zero homomorphisms from A into \mathbb{C} and A^* be the dual space of A consisting of all bounded linear functions from A into \mathbb{C} .

Throughout this paper, we assume that A is a Banach algebra such that $\Delta(A) \neq \emptyset$.

Let $\{e_\alpha\}$ be a net in a Banach algebra A . The bounded net $\{e_\alpha\}$ is called,

1. a *bounded approximate identity* (BAI) if, for each $a \in A$, $\|ae_\alpha - a\| + \|e_\alpha a - a\| \rightarrow 0$,
2. a *bounded weak approximate identity* (BWAI) if, for each $a \in A$, $|\varphi(e_\alpha a) - \varphi(a)| \rightarrow 0$ for all $\varphi \in \Delta(A)$ [4].

Suppose that $S = \{m/n : m, n \in \mathbb{Z}^+\}$ where \mathbb{Z}^+ shows the set of positive integers. Clearly, S is a cancellative semigroup under addition. So, $A = \ell^1(S)$ is a semi-simple semigroup algebra which has a weak approximate identity of norm one, but it does not have any bounded or unbounded approximate identity, the proof of this fact is almost complicated; see [4] and references there in for more details.

This example is a motivation of presenting the concept of bounded weak approximate identity and shows that two concepts of (bounded) approximate identity and (bounded) weak approximate identity are different.

3 Results and applications

After Jones and Lahr, the first application of the notion of BWAI appeared in a paper of Takahasi and Hatori [20] as follows.

Suppose that A is a commutative Banach algebra. We say that A is without order if for $a \in A$, the condition $aA = \{0\}$ implies $a = 0$ or equivalently A does not have any non-zero annihilator. For example if A has an approximate identity, then it is without order. A linear operator T on A is called a multiplier if it satisfies $aT(b) = T(a)b$ for all $a, b \in A$. Suppose that $\mathcal{M}(A)$ denotes the space of all multipliers of the Banach algebra A . For each $T \in \mathcal{M}(A)$, there exists a unique bounded continuous function \widehat{T} on $\Delta(A)$ such that $\widehat{T(a)}(\phi) = \widehat{T}(\phi)\widehat{a}(\phi)$ for all $a \in A$ and $\phi \in \Delta(A)$; see [15, Proposition 2.2.16]. Let $\widehat{\mathcal{M}}(A)$ denote the space of all \widehat{T} corresponding to $T \in \mathcal{M}(A)$.

A bounded continuous function σ on $\Delta(A)$ is called a BSE-function if there exists a constant $C > 0$ such that for each $\phi_1, \dots, \phi_n \in \Delta(A)$ and complex numbers c_1, \dots, c_n , the inequality

$$\left| \sum_{i=1}^n c_i \sigma(\phi_i) \right| \leq C \left\| \sum_{i=1}^n c_i \phi_i \right\|_{A^*}$$

holds. Let $C_{\text{BSE}}(\Delta(A))$ be the set of all BSE-functions.

Definition 1 A without order commutative Banach algebra A is called a BSE-algebra if

$$C_{\text{BSE}}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

Theorem 1 Let A be a commutative and without order Banach algebra. Then $\widehat{\mathcal{M}(A)} \subseteq C_{\text{BSE}}(\Delta(A))$ if and only if A has a BWAI.

Proof. See [20, Corollary 5]. \square

For a large class of Segal algebras, BSE-property characterized exactly through the existing of a BWAI. Recall that if G is a locally compact group, a subspace \mathcal{S} of $L^1(G)$ is called a Segal algebra if it satisfies the following conditions:

1. \mathcal{S} is dense in $L^1(G)$.
2. \mathcal{S} is a Banach space under some norm $\|\cdot\|_{\mathcal{S}}$ such that $\|f\|_1 \leq \|f\|_{\mathcal{S}}$ for each $f \in \mathcal{S}$.
3. $L_y f$ is in \mathcal{S} and $\|f\|_{\mathcal{S}} = \|L_y f\|_{\mathcal{S}}$ for all $f \in \mathcal{S}$ and $y \in G$ where $L_y f(x) = f(y^{-1}x)$.
4. For all $f \in \mathcal{S}$, the mapping $y \rightarrow L_y f$ is continuous.

Theorem 2 Let G be a locally compact abelian group and $S(G)$ be a Segal algebra. Then $S(G)$ is a BSE-algebra if and only if it has a BWAI.

Proof. See [9, Lemma 1.1] \square

Kamali and Bami generalized the above result to essential abstract Segal algebra as follows [21]. Recall that a Banach algebra B is an abstract Segal algebra of a Banach algebra A if

1. B is a dense left ideal in A .
2. there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for each $b \in B$.
3. there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for each $a, b \in B$.

Also, we say that B is essential in A if B is an ideal in A and

$$B = \{ax : a \in A, x \in B\}.$$

Theorem 3 Let $(A, \|\cdot\|_A)$ be a BSE-algebra and $(B, \|\cdot\|_B)$ be an essential abstract subalgebra of A . Then B is a BSE-algebra if and only if it has a BWAI.

Proof. See [21, Theorem 3.1]. \square

There is a variant of Theorem 3 such that has been proved by the author and Nemati in [16].

One of the notable works after the paper of Takahasi and Hatori is the paper of Kaniuth and Ülger. The following theorem gives a nice application of BWAI.

Theorem 4 *Let A be a semi-simple commutative Banach algebra which is an ideal in its second dual. then the following assertions are equivalent.*

1. A is a BSE-algebra.
2. A has a BWAI.
3. A has a BAI.

Proof. See [5, Theorem 3.1]. □

In general, the notion of BWAI is a fundamental tool in studying the BSE-algebras.

3.1 Bounded weak approximate identities with respect to a subset of A^*

This subsection is a short part of the paper [7]. Suppose that A is a Banach algebra and $E \subseteq A^* \setminus \Delta(A)$. A complex-valued bounded continuous function σ on $\Delta(A) \cup E$ is called a BSE-like function if there exists an $M > 0$ such that for each $f_1, f_2, f_3, \dots, f_n \in \Delta(A) \cup E$ and complex numbers $c_1, c_2, c_3, \dots, c_n$,

$$\left| \sum_{i=1}^n c_i \sigma(f_i) \right| \leq M \left\| \sum_{i=1}^n c_i f_i \right\|_{A^*}. \quad (1)$$

We show the set of all the BSE-like functions on $\Delta(A) \cup E$ by $C_{\text{BSE}}(\Delta(A), E)$ and let $\|\sigma\|_{\text{BSE}}$ be the infimum of all M satisfying relation 1. Obviously, $C_{\text{BSE}}(\Delta(A), E)$ is a linear subspace of $C_b(\Delta(A) \cup E)$ and we have

$$\{\sigma|_{\Delta(A)} : \sigma \in C_{\text{BSE}}(\Delta(A), E)\} \subseteq C_{\text{BSE}}(\Delta(A)).$$

Definition 2 We say that A has a BWAI with respect to E if there exists a bounded net $\{x_\alpha\}$ in A with

$$\lim_{\alpha} f(x_\alpha) = 1 \quad (f \in \Delta(A) \cup E).$$

Using [7, Theorem 4.2], one can check that $1 \in C_{\text{BSE}}(\Delta(A), E)$ if and only if A has a BWAI with respect to E .

Clearly, every BWAI with respect to E is a BWAI but the converse is not valid in general unless $E = \emptyset$. To see an example, take $A = \mathbb{R}$ and $E = \{f_2\}$ where $f_2(x) = 2x$ for all $x \in \mathbb{R}$. Now, it is clear that the bounded sequence $\{\frac{n}{n+1}\}$ is a BWAI for \mathbb{R} but it is not a BWAI with respect to E . Note that for $A = \mathbb{R}$, we have $\Delta(A) = \{f_1\}$. Also, to see an example of a non-trivial BWAI respect to E , take $A = C([0, 1])$ and $E = \{F_{dx}\}$ where $F_{dx}(f) = \int_0^1 f(x)dx$. Now it is clear that the bounded sequence $\{f_n = \frac{x^2+n}{n}\}$ is a BWAI with respect to E .

To see another variant of BWAI see a recent work by Farrokhzad and the author of the paper in [17].

3.2 Amenability and bounded weak approximate identities

Forrest and Skantharajah found another application of BWAI. Indeed, it is shown that for a large class of groups G , an ideal I of $A(G)$; the Fourier algebra, has a BAI if and only if it has a BWAI.

Theorem 5 *suppose that G is an amenable [SIN]-group. Then the following assertions are equivalent if I is a closed ideal of $A(G)$.*

1. I has a BWAI.
2. I has a BAI.
3. $Z(I)$ is in $\mathfrak{R}_c(G)$.

Proof. See [2, Corollary 2]. □

Note that $Z(I) = \{x \in G : u(x) = 0 \text{ for every } u \in I\}$ is in $\mathfrak{R}_c(G)$ if $Z(I)$ is closed and it is an element of $\mathfrak{R}(G_d)$; the ring of subsets of G generated by all left cosets of subgroups of G_d , where G_d denotes G with the discrete topology.

Suppose that G is a locally compact group. The group G is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$ where $L_x f(y) = f(x^{-1}y)$ and $L^\infty(G)$ is the space of all measurable essentially bounded functions (equivalent classes) from G into \mathbb{C} ; see [18, Definition 4.2].

Using the notion of BWAI one can characterize the amenability of group G as the next three results show.

Theorem 6 *Let G be a discrete group. Then G is amenable if and only if $A(G)$ has a BWAI.*

Proof. See the last paragraph in [2]. □

A better version of the above result obtained by Kaniuth and Ülger to the best of our knowledge.

Theorem 7 *Let G be a locally compact group. Then G is amenable if and only if $A(G)$ has a BWAI.*

Proof. See [5, Theorem 5.1] or [8, Proposition 3.11]. □

For $1 < p < \infty$ there is a version for the *Figà Talamanca-Herz algebra* $A_p(G)$ of Theorem 7; see [14, Theorem 2].

Let E be a closed non-empty subset of G and $1 < p < \infty$. Define

$$Z_p(E) = \{u \in A_p(G) : u(x) = 0 \text{ for all } x \in E\}.$$

Theorem 8 *Let G be a locally compact group. Then G is amenable if and only if there exists a proper closed subgroup H of G such that $I_p(H)$ has a BWAI.*

Proof. See [14, Theorem 5]. □

The above theorem gives the converse of a theorem due to Forrest, Kaniuth, Lau, and Spronk [3, Corollary 4.2]. For more details see [14, Corollary 2].

Laali and the author of the paper in [13], introduced a variant of character amenability using the BWAI.

Definition 3 Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. We say that A is Δ -weak φ -amenable, if there exists an $m \in A^{**}$ such that $m(\varphi) = 0$ and $m(\psi.a) = \psi(a)$ for each $a \in \ker(\varphi)$ and $\psi \in \Delta(A)$, where $\ker(\varphi) = \{x \in A : \varphi(x) = 0\}$.

The above definition characterized with use of BWAI as follows.

Theorem 9 Let A be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. Then A is Δ -weak φ -amenable if and only if $\ker(\varphi)$ has a BWAI

Proof. See [13, Theorem 2.1]. □

We end this paper with the following questions which up to our best knowledge they are still open.

1. Ghahramani and Lau in [6] introduced and studied a new closed ideal of $A(G)$. Indeed, let G be a locally compact group and put

$$\mathfrak{L}A(G) = L^1(G) \cap A(G)$$

with the norm

$$\|f\| = \|f\|_1 + \|f\|_{A(G)}.$$

Clearly $\mathfrak{L}A(G)$ with pointwise multiplication is a commutative Banach algebra such that $\Delta(\mathfrak{L}A(G)) = G$ and it is called the *Lebesgue-Fourier algebra* of G . Does the amenability of G imply the existence of a BWAI for $\mathfrak{L}A(G)$?

2. Runde in [19], by using the canonical operator space structure of $L^p(G)$, introduced and studied the algebra $OA_p(G)$ for $1 < p < \infty$, the operator Figà Talamanca-Herz algebra. It was shown that $A_p(G) \subseteq OA_p(G)$ [19, Remark 4. pp. 159] and $OA_p(G)$ has a BAI if and only if G is an amenable group [19, Theorem 4.10]. Is it true that Theorem 7 remains correct if we replace $A_p(G)$ by $OA_p(G)$? Note that this theorem has a version for $A_p(G)$.
3. In [13, Theorem 4.1 (1)], we show that if G is an amenable group, then $L^1(G)$ is Δ -weak φ -amenable for each $\varphi \in \Delta(L^1(G)) \cup \{0\}$. Is the converse of this assertion true?
4. We know that if \mathbb{G} is a locally compact quantum group as it is introduced in [12] and $L^1(\mathbb{G})$ denotes the related group algebra, generally, $L^1(\mathbb{G})$ does not have any bounded approximate identity. On the other hand, if we focus on the quantum group $\mathbb{G}_s = VN(G)$ where G is not amenable, then one can see that $L^1(\mathbb{G}) = A(G)$ does not have a BWAI. Therefore, in general $L^1(\mathbb{G})$

does not have any BWAI. But here we can raise the following question. Is the implication

$$L^1(\mathbb{G}) \text{ has a BWAI only if it has a BAI}$$

true?

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