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Arens Regularity and Cohomological Properties of Banach Lattice Algebra

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Abstract A Banach lattice algebra is a Banach lattice, an associative algebra with a sub-multiplicative norm and the product of positive elements should be positive. In this note we study the Arens regularity and cohomological properties of Banach lattice algebras.

Keywords 1-Banach lattice algebra · Arens regularity · Derivation

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1 Introduction

A.W. Wickstead in [6] has been introduced the Banach lattice algebra. A Banach lattice E with associative algebra and a sub-multiplicative norm is a Banach lattice algebra whenever the product of positive elements is should be positive element. For example, if X is a locally compact Hausdorff space then $C_0(X)$, the continuous scalar functions which vanish at infinity, is a Banach

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space under the supremum norm, and a Banach lattice with pointwise order. It is a Banach algebra under the pointwise multiplication. If X is compact then the identity (the constantly one function) has norm one. Let A be a Banach lattice alebra. If there is an identity which has norm one then A is said to be a $1-$Banach lattice algebra. The second dual A^{**} of Banach algebra A endowed with the Arens multiplications is a Banach algebra, see [2]. The constructions of the two Arens multiplications in A^{**} lead us to define topological centers for A^{**} with respect to both Arens multiplications, for details see [3,5]. In this note, we will study the Arens regularity and cohomological properties of Banach lattice algebras.To state our results, we need to fix some notation and recall some definitions. A Banach lattice E is said to be KB-space whenever each increasing norm bounded sequence of E^+ is norm convergent. A subset A of a Riesz space E is called b-order bounded in E if it is order bounded in the bidual $E^{\sim\sim}$. A Riesz space E is said to have property (b) if $A \subset E$ is order bounded whenever A is order bonded in $E^{\sim\sim}$. Note that every perfect Riesz space and therefor every order dual and every reflexive Banach lattice has property (b). Let E be a Riesz space. A linear functional $f : E \to \mathbb{R}$ is said to be positive whenever $f(x) \geq 0$ holds for each $x \in E^+ = \{x \in E : x \geq 0\}$. A linear functional f is called order bounded if it maps order bounded subsets of E onto bounded subsets of $\mathbb{R}$. It is clear that every positive operator is order bounded, but the converse is not true in general. In all undefined terminology concerning Riesz spaces, Arens regularity, we will adhere to [1,3].

2 Mine result

Suppose that A is a Banach algebra and A^*, A^{**}, respectively, are the first and second duals of A. We define the first and second topological center of A^{**}, which are

$$Z_1(A^{**}) = Z_{A^{**}}(A^{**}) = \{a'' \in A^{**} : b'' \to a''b'' \textit{ is weak}^*\textit{−weak}^* \textit{ continuous}\},$$

$$Z_2(A^{**}) = Z^t_{A^{**}}(A^{**}) = \{a'' \in A^{**} : a'' \to a''ob'' \textit{ is weak}^*\textit{−weak}^* \textit{ continuous}\},$$

where $a''b''$ and $a''ob''$ are the first and second Arens product in A^{**}, respectively the Banach algebra A is called Arens regular, if $Z_1(A^{**}) = A^{**}$. In [5], Lau and Ulger proved that a weakly sequentially complete Banach algebra A with a sequentially bounded approximate identity can not be Arens regular unless it is unital.

Theorem 1 *Let E be a $1-$Banach lattice algebra which is a KB-space. Then E is Arens regular.*

Proof. Since every KB-space is weakly sequentially complete and E is unital, by [5, Corollary 2.7], E is Arens regular. □

Note: Let $E_1, E_2, ...E_n$ be Banach lattices with norms $\|.\|_i$ and orders $\leq_i$ for $1 \leq i \leq n$. Then $\oplus_{i=1}^n E_i$, with norm $\|x\| = \Sigma_{i=1}^n \|x_i\|_i$ and under the order $\{x_i\}_{i=1}^n \leq \{y_i\}_{i=1}^n$ whenever $x_i \leq_i y_i$ holds for all $1 \leq i \leq n$, is a Banach lattice. It is clear that the following assertions are hold.

(a) $E_1, E_2, ...E_n$ are KB-spaces if and only if $\oplus_{i=1}^n E_i$ is KB-space.
(b) If $E_1, E_2, ...E_n$ are Banach lattice algebras, then $\oplus_{i=1}^n E_i$ is a Banach lattice algebra.
(c) If $E_1, E_2, ...E_n$ are Arens regular Banach algebras, then $\oplus_{i=1}^n E_i$ is Arens regular Banach algebra.

By [3,Example 2.6.22], we know that $(\ell^1, .)$ with pointwise multiplication is Arens regular. Now, by following example we see another proof for Arens regularity of $(\ell^1, .)$.

Example 1 (a) Every perfect $1-$Banach lattice algebra is Arens regular.
(b) If E is a KB-space, then $E \oplus \mathbb{R}$ is a KB-space, therefore it is Arens regular.
(c) Since $(\ell^1, .)$ is a KB-space, $(\ell^1 \oplus \mathbb{R}, .)$ is Arens regular, and so $(\ell^1, .)$ is Arens regular.

Theorem 2 *Let E be a $1-$Banach lattice algebra with property (b). If E has order continuous norm, then E is Arens regular.*

Proof. Since E has property (b) and its norm is order continuous, E is a $KB-$space. So, by [1, Theorem 4.60], E is weakly sequentially complete. Since E is unital, by [5, Corollary 2.7], E is Arens regular. □

Now by using [2, Proposition 2.10], we have the following result.

Corollary 1 *Let E be a $1-$Banach lattice algebra. If the identity operator $I : E \to E$ is $b-$weakly compact, then E is Arens regular.*

The systematic study of the notion of amenability has its origin in the beginning of the modern measure theory in the earlier part of the twentieth century. It is worthwhile to say that the theory of amenable Banach algebras begins from B. E. Johnson's memoir [4], and the choice of terminology comes from Theorem 2.5 in [4]. The purpose of this note is to give an overview of what has been done so far on semi-amenability and various notions of semi-amenability and raise some problems in semi-amenability and Connes semi-amenability of Banach algebras.
First, we recall some standard notions; for further details, see [3].
Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule. A bounded linear mapping $\mathcal{D} : \mathcal{A} \to \mathcal{X}$ is a derivation if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$, for all $a, b \in \mathcal{A}$. For each $x \in \mathcal{X}$, the mapping $\delta_x(a) = ax - xa$, $(a \in \mathcal{A})$ is a continuous derivation, called an inner derivation. A Banach algebra $\mathcal{A}$ is called amenable (resp. contractible) if for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$ every continuous derivation $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{X}^*$ (resp.into $\mathcal{X}$) is inner i.e. $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*) = \{0\}$ (resp.$\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = \{0\}$), where $\mathcal{H}^1(\mathcal{A}, \mathcal{X}^*)$ is the first cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}^*$. If $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$, then A is called weakly amenable.

Definition 1 A Banach algebra $\mathcal{A}$ is called supper amenable if $\mathcal{H}^1(\mathcal{A}, E) = \{0\}$ for every Banach $\mathcal{A}$-bimodule E.

Definition 2 If $\mathcal{D} : \mathcal{A} \to E$ is a positive derivation, then $\mathcal{D}x \geq 0$ whenever $x \geq 0$.

Theorem 3 *Let $\mathcal{A}$ be supper amenable, then there is not positive derivation $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{A}$, $\mathcal{H}^1(\mathcal{A}, \mathcal{A})^+ = \{0\}$.*

Proof. If $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a non-zero positive derivation, then there is $x \in \mathcal{A}$, $\mathcal{D}(a) = id_x(a) > 0$ which follows that $ax - xa > 0$ for each $a \in \mathcal{A}$. It follows $x^2 > x^2$ which is impossible. □

Corollary 2 *If there exists a positive derivation from $\mathcal{A}$ into $\mathcal{A}$, then $\mathcal{A}$ is not super amenable.*

Let $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ be a derivation. If $\mathcal{D}^+$ exists as derivation, then $\mathcal{A}$ is not super amenable.

Theorem 4 *Let $\mathcal{A}$ be a Banach lattice algebra and there is positive derivation $\mathcal{D} : \mathcal{A} \to \mathcal{A}^*$ .Then $\mathcal{A}$ is not weakly amenable.*

Proof. $\mathcal{D} > 0$ and $\mathcal{D} = id_{a'}$ for some $a' \in \mathcal{A}^*$. It follows that $a'a - aa' > 0$, therefore $a'(x)a - aa'(x) > 0$ for each $x \in \mathcal{A}$, then $a > a$ which is impossible. □

$\mathcal{A}$ is not amenable, because $\mathcal{D} > 0$ and $\mathcal{D} = id_{x'}$, $x'a - ax' > 0$ then $x'(x)a - ax'(x) > 0$ for all $x \in X$ thus $a > a$ which is impossible.

Theorem 5 *Let $\mathcal{A}$ be a Dedekind Banach lattice algebra and $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ be an order bounded derivation. If for each $a, b \in \mathcal{A}$ and $0 \leq z \leq ab$, there exists $0 \leq x \leq a$, $0 \leq y \leq b$ with $xy = z$. Then $\mathcal{A}$ is not super amenable.*

Proof. By Theorem 1.18 of [1], $\mathcal{D}^+$ exits. We show that $\mathcal{D}^+$ is a derivation. Let $0 \leq x \leq a$ and $0 \leq y \leq b$. Then $0 \leq xy \leq ab$ and we have

$$\mathcal{D}xy \leq \mathcal{D}^+ab \Rightarrow x\mathcal{D}y + y\mathcal{D}x \leq \mathcal{D}^+ab \Rightarrow a\mathcal{D}^+b + b\mathcal{D}^+a \leq \mathcal{D}^+ab, \tag{1}$$

if $0 \leq z \leq ab$ then there are $0 \leq x \leq a$, $0 \leq y \leq b$ with $xy = z$ where

$$\mathcal{D}z = \mathcal{D}xy = x\mathcal{D}y + y\mathcal{D}x \leq a\mathcal{D}^+b + b\mathcal{D}^+a \Rightarrow \mathcal{D}^+ab \leq a\mathcal{D}^+b + b\mathcal{D}^+a, \tag{2}$$

from (1) and (2) it follow that $\mathcal{D}^+ab = a\mathcal{D}^+b + b\mathcal{D}^+a$. By contradiction, let $\mathcal{A}$ be super amenable. □

For example $L^1(\mathbb{R})$ is not super amenable.

Theorem 6 *Let $\mathcal{A}$ be a Banach lattice algebra with multiplicative identity e. If e is order unit for $\mathcal{A}$, then there is not positive derivation from $\mathcal{A}$ into $\mathcal{A}$.*

Proof. By contradiction, let $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ be a positive derivation. Then $\mathcal{D}e = 0$. Since e is order unit, for each $0 < x \in \mathcal{A}^+$ there is $\lambda > 0$ with $\lambda e > x$. It follow that $0 = \lambda\mathcal{D}e = \mathcal{D}(\lambda e) > \mathcal{D}x$ which is impossible. □

Example 2 There is not positive derivation from l^∞ into l^∞.

Let $\mathcal{A}$ be a Banach lattice algebra and $a, b \in \mathcal{A}^{+}$. $\mathcal{A}$ has property decomposition, if for each $0 \leq z \leq ab$ there are $0 \leq x \leq a$ and $0 \leq y \leq b$ with $z = xy$. [PD] Let $\mathcal{A}$ be the Banach lattice $(X = \mathcal{A})$-bimodule whose underlying space is X, but on which X acts via $\pi_l(a, x) = a.x := ax$ and $x.a := 0\,(x \in X, a \in \mathcal{A})$. If $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{D}(a) = a$. Then $\mathcal{D}$ is a derivation and positive,

$$\mathcal{D}(ab) = a\mathcal{D}b + (\mathcal{D}a)\,b = a\mathcal{D}b = ab \Rightarrow \mathcal{D}(ab) = ab = \pi_l(a,b) + \pi_r(a,b) = a \cdot \mathcal{D}b + (\mathcal{D}a) \cdot b. \quad (3)$$

Let $\alpha_0 \leq x \leq \beta$, therefore $\alpha_0 e_\alpha \leq x e_\alpha \leq \beta e_\alpha$ and hence $\alpha_0 e_\alpha \leq \mathcal{D}(x e_\alpha) \leq \beta e_\alpha$, it follow that $\alpha_0 \leq \mathcal{D}x \leq \beta$. Every Banach lattice algebra is not super amenable.

Theorem 7 *Let $\mathcal{A}$ be a unital Banach lattice algebra. There is a positive bounded derivation $\mathcal{D} : \mathcal{A} \to X$ (for some $\mathcal{A}$-module X) which is inner.*

Proof. Let $X := \mathcal{A}$ be the Banach $\mathcal{A}$-bimodule whose underlying space is $\mathcal{A}$, but on which acts via $x \cdot a = xa$ and $a \cdot x = 0\,(a \in \mathcal{A}, x \in X)$. Set $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{D}(a) = a$ for all $a \in \mathcal{A}$. Then $\mathcal{D}$ is a positive derivation, by following equalities $\mathcal{D}(a) \cdot b + a \cdot \mathcal{D}b = ab = \mathcal{D}(ab)$ for each $a, b \in \mathcal{A}$. Let e be a unit multiplication for $\mathcal{A}$. Then set $\mathcal{D} := id_e$ (where $e \in X$), that is for all $a \in \mathcal{A}$ we have $\mathcal{D}(a) = id_e(a) = e \cdot a - a \cdot e = ea = a$. □

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