Global Analysis and Discrete Mathematics
Volume 3, Issue 1, pp. 33–52
ISSN: 2476-5341

Finite Time Mix Synchronization of Delay Fractional-Order Chaotic Systems

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Received: 10 October 2017 / Accepted: 27 May 2018

Abstract Chaos synchronization of coupled fractional order differential equation is receiving increasing attention because of its potential applications in secure communications and control processing. The aim of this paper is synchronization between two identical or different delay fractional-order chaotic systems in finite time. At first, the predictor-corrector method is used to obtain the solutions of delay fractional differential equations in discrete times. The mix synchronization problem is formulated as an optimization problem. A modified version of particle swarm optimization (MPSO) method is used for solving the problem. It is shown that the proposed method can be applied in wide range of master-slave systems with commensurate or non-commensurate fractional orders. Numerical simulations show the efficiency and robustness of the proposed method.

Keywords Synchronization · Fractional-order delayed chaotic system · Particle swarm optimization

Mathematics Subject Classification (2010) 90C59 · 90C90 · 34H1 · 34A08

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1 Introduction

The concept of a dynamical system has its origin in Newtonian mechanics. Since the first chaotic model was found by Lorenz in 1963 [19], researchers have laid themselves out to construct new chaotic systems and analyze their dynamical behaviors [20]. In this paper, a chaotic system is a deterministic nonlinear dynamical system that possesses some complex dynamical behaviors, such as extreme sensitivity to initial conditions. This means that two systems starting trajectories from their arbitrary and almost the same initial states could evolve in dramatically different fashions, and soon become uncorrelated and unpredictable. Despite its complexity and unpredictability, chaos can be controlled and two chaotic systems can be synchronized [1,21].

The fractional calculus started from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), and it has been developed progressively up to now. During the last few decades, fractional calculus has become a powerful tool in describing the dynamics of complex systems which appear frequently in several branches of science and engineering. Insofar as it concerns the applications of fractional derivatives, we can cite researches in different areas such as viscoelastic damping, anomalous diffusion process, signal processing, electrochemistry and fluid flow [7]. There are amount of efforts to discover the chaos of fractional order systems. Specially, chaotic features have been proofed in fractional order Lorenz system, fractional order Chua's system, fractional order Dung's oscillator, fractional order Genesio-Tesi system and fractional order Lotka-Volterra system [4]. Recent investigations in physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using fractional differential equations with time delay [3,14]. Moreover, some Fractional-order Delay Differential Equations (FDDE) have chaotic behaviors [26,29,30]. In recent years, FDDE begin to arouse the attention of a number of researchers. Recently, Pandey et. al., investigates approximation method for solving FDDE [25]. Finite time stability analysis of FDDE is done by Li and Wang [15]. One can find further subjects in this area in [23]. Synchronization, which means "things occur at the same time or operate in unison", has been a subject of great interest and importance, in theory but also various fields of application, such as secure communication [12] and neuroscience [24]. There exist some various types of synchronization. Now, some basic definitions about synchronization are given. Let $x(t)$, $y(t)$ and $\|.\|$ denote dynamics of master system, slave system and norm(usually the L^2 norm) of a vector respectively.

Definition 1 The master-slave system $(x(t), y(t))$ is said to completely synchronized in time τ if

$$\|x(t) - y(t)\| = 0, \ t \in (\tau, \infty). \tag{1}$$

Definition 2 The master-slave system $(x(t), y(t))$ is said to anti-synchronized in time τ if

$$\|x(t) + y(t)\| = 0, \ t \in (\tau, \infty). \tag{2}$$

Definition 3 The master-slave system $(x(t), y(t))$ is said to reach η-lag synchronization in time τ if

$$\|x(t) - y(t + \eta)\| = 0, \ t \in (\tau, \infty), \tag{3}$$

where η is given constant.

Many approaches have been presented for the control and synchronization of chaotic systems such as impulsive control [28], fuzzy sliding mode control [16] and adaptive control [18]. Moreover, in the past few decades, several intelligent optimization algorithms have been widely applied to control and synchronization of chaotic systems, such as optimization strategy [31], differential evolution algorithm (DE) [17], particle swarm optimization (PSO) [5] and artificial bee colony algorithm (ABC) [10]. Synchronization with using an iteration algorithm is easy, fast and does not have complexity and difficulty of analytical methods. In this method, first the synchronization problem is transformed into an optimization problem and then it is solved with an intelligent optimization methods. However, many of them have the drawbacks of premature convergence, low searching accuracy and iterative inefficiency. Especially the problems involving multiple peak values fall into local optima since the search space of the problem increases at each iteration. In this paper, a modified version of PSO algorithm is used to overcome theses difficulties. The remainder of this paper is organized as follows. In Section 2, fractional derivatives and predictor-corrector scheme for solving FDDE are introduced. In Section 3, the synchronization of two chaotic systems is explained. This problem will be formulated as an optimization problem. Section 4 is devoted to PSO algorithm. In this section some modifications on PSO algorithm are proposed. The proposed method is applied on Logistic, Lorenz, Liu and Chen systems. Numerical results are given in Section 5. In Section 6, a conclusion is drawn.

2 Preliminaries

There are several definitions of fractional derivatives, such as Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions [14]. In this section, an introduction of fractional derivatives will be given.

Definition 4 A real function $f(t)$, $t > 0$ is said to be in space C_α, $\alpha \in \Re$ if there exists a real number $p(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C\,[0, \infty]$.

Definition 5 A real function $f(t), t > 0$ is said to be in space C_α^m, $m \in N \cup \{0\}$ if $f^{(m)}(t) \in C_\alpha$.

Definition 6 The Riemann-Liouville integral operator of order β is defined as

$$J^\beta z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} z(\tau) d\tau, \tag{4}$$

where $\Gamma(.)$ is the gamma function.

Definition 7 The Riemann-Liouville fractional derivative of $y(t)$, $y(t) \in C^m_{-1}$, $m \in N \cup \{0\}$, is defined as

$$ {}_L D^{\alpha} y(t) = D^{(m)}(J^{m-\alpha})y(t) = \frac{1}{\Gamma(m-\alpha)}(\frac{d}{dx})^n \int_0^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau, \quad (5) $$

where $\alpha > 0$, $m-1 < \alpha < m$ $(m \in N)$, $D^{(m)}$ is the ordinary mth derivative of y, J is the Riemann-Liouville integral operator.

Definition 8 The Caputo fractional derivative of $y(t)$, $y(t) \in C^m_{-1}$, $m \in N \cup \{0\}$, is defined as

$$ D^{\alpha} y(t) = J^{m-\alpha} y^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau, \quad (6) $$

where $\alpha > 0$, $m-1 < \alpha < m$ $(m \in N)$, $y^{(m)}$ is the ordinary mth derivative of y, J is the Riemann-Liouville integral operator.

Notice in many physical systems, system components are forced into some configurations, or initialized. Fractional order components, however, require a more complete initialization, as they have an inherent time-varying memory effect. We assume that fractional order integral is beginning at time $t = 0$. Note that for $m-1 < \alpha \leq m$ $(m \in N)$,

$$ J^{\alpha} D^{\alpha} y(t) = y(t) - \sum_{k=0}^{m-1} \frac{d^k y}{dt^k}(0) \frac{t^k}{k!}, \quad (7) $$

and for $\alpha \to m$, the Caputo derivative becomes a conventional mth derivative. Based on the expression (6), it is recognized that the Caputo derivative has a relationship with all of the function history information. However, the ordinary derivative is only related to its nearby points. Thus, a model described by fractional-order differential equations possesses memory. In addition, the initial conditions of Caputo derivative differential equations take on the same form as for integer-order ones, which have well understood features of physical situations and more applications in real world problems. Moreover, Heymans and Podlubny show that it is possible to attribute physical meaning to initial conditions expressed in term of Riemann-Liouville fractional derivatives [11]. Caputo derivative and Riemann-Liouville derivative are connected with each other by the following relation [8]

$$ (D^{\alpha} y)(t) = ({}_L D^{\alpha} y)(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{\Gamma(k-\alpha+1)} (t)^{k-\alpha}. \quad (8) $$

2.1 Predictor-corrector scheme of fractional-order delayed equation

Predictor-corrector method is a numerical algorithm for solving ordinary differential equations. Diethem et al. modified this algorithm for applying on fractional differential equation [9]. Bhalekar and Daftardar-Gejji [2] improve it to solve FDDE. Consider the following FDDE:

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t), y(t-\tau), y_0), t \in [0, T], 0 < \alpha \leq 1; \\ y(t) = g(t), \quad t \in [-\tau, 0] \end{cases} \tag{9}$$

where f is in general a nonlinear function. We consider the following uniform grid

$$T_N = \{t_n = nh : n = -k, -k+1, \cdots, -1, 0, 1, \cdots, N\}, \tag{10}$$

where k and N are integers such that $h = \frac{T}{N} = \frac{\tau}{k}$. Let

$$y_h(t_j) = g(t_j), \quad j = -k, -k+1, \cdots, -1, 0, \tag{11}$$

and

$$y_h(t_j - \tau) = y_h(jh - kh) = y_h(t_{j-k}), \ j = 0, 1, \cdots, N. \tag{12}$$

It is assumed that the approximated solution

$$y_h(t_j) \approx y(t_j), \ j = -k, -k+1, \cdots, -1, 0, 1, \cdots, n,$$

is obtained and the value of $y_h(t_{n+1})$ should be calculated. If the operator $J^{\alpha}_{t_n+1}$ is applied on the both sides of (6), we have

$$y(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \xi)^{\alpha-1} \times f(\xi, y(\xi), y(\xi - \tau)) d\xi. \tag{13}$$

The approximations $y_h(t_n)$ is used for $y(t_n)$ in (13). The trapezoidal quadrature formula is used for approximating (13). Therefore,

$$y_h(t_{n+1}) = g(0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1}), y_h(t_{n+1-k})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \times \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \tag{14}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-a)(n+1)^{\alpha}, \quad j = 0; \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, \ 1 \leq j \leq n; \\ 1, \quad j = n+1. \end{cases}$$

The predictor term $y_h^p(t_{n+1})$ is calculated as follows

$$y_h^p(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \Sigma_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \tag{15}$$

where

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha}). \tag{16}$$

3 Problem formulation

Consider the following $\mathbb{k}$-dimensional fractional-order chaotic delayed systems:

$$\begin{cases} D^{\alpha}y(t) = f(y(t), y(t-\tau), y_0) + u(t) \\ D^{\beta}x(t) = g(x(t), x(t-\tau), x_0), \end{cases} \tag{17}$$

where f and g are in general a nonlinear function of its arguments, D^{α} and D^{β} denote fractional derivatives of order α and β respectively and $(\alpha, \beta) \in (0,1)^2$. Let $u(t) = (u_1(t), u_2(t), \cdots, u_{\mathbb{k}}(t))^t$ be a local controller defined between initial time $t_0 = 0$ and given final time T. We suppose that two chaotic systems (17) have different initial conditions $(x_0 \neq y_0)$. Our aim is synchronizing two coupled chaotic systems (17) by applying a suitable control function $u(t)$. We assume that the control function u_j is restricted to given constant values A_j and B_j for $j = 1, \cdots \mathbb{k}$. Therefore, the synchronization problem of system (17) is formulated as follows

$$\begin{aligned} &\min \ \ \|X(x(t)) - Y(y(t))\| \\ &s.t. \ \ A_j \leq u_j(t) \leq B_j, \ t \in (0,T), \ j = 1, \cdots, \mathbb{k} \end{aligned} \tag{18}$$

where X, Y are functions that declare type of synchronization and $\|.\|$ denotes the $L^2(0,T)$-norm. The predictor-corrector method is used for solving (17). We consider approximated solution of (17) in discrete times defined in (10). Therefore, the optimization problem (18) can be presented as follows

$$\begin{aligned} &\min \ \ \left(\sum_{i=1}^{N} |X(x(t_i)) - Y(y(t_i))|^2 \right)^{\frac{1}{2}} \\ &s.t. \ \ A_j \leq u_j(t_i) \leq B_j, \ i = 1, \cdots, N, \ j = 1, \cdots, \mathbb{k} \end{aligned} \tag{19}$$

where $|\ .\ |$ denotes the Euclid norm.

3.1 PSO and modification PSO

Particle swarm optimization (PSO) is a population based stochastic optimization technique developed by R. C. Eberhart and J. Kennedy in 1995, inspired by social behavior of bird flocking or fish schooling [13]. In PSO, candidate solutions of a specific optimization problem are called particles. Suppose that the search space is Ω-dimensional, each particle in the searching space is characterized by two factors, i.e., position $X_0 = (x_{i1}, x_{i2}, \cdots, x_{i\Omega})$ and velocity $V_0 = (v_{i1}, v_{i2}, \cdots, v_{i\Omega})$, where i denote the ith particle in the swarm. The fitness of each particle can be evaluated according to the objective function of optimization problem. PSO starts with the random initialization of a swarm of particles in the search space. Let $pbest_i(k)$ be the best position found by particle i within k iteration steps. Let $gbest(k)$ denotes the best position within k iteration steps for all particles. All particles update their velocities and positions based on their own experience $pbest_i(k)$ and experience of all particles

$gbest(k)$. The updating rules of velocity and position are given by (20) and (21) respectively:

$$v_i^{k+1} = wv_i^k + c_1 r_1 (pbest_i(k) - x_i^k) + c_2 r_2 (gbest(k) - x_i^k),\ i = 1, \cdots, \Omega, \quad (20)$$
$$x_i^{k+1} = x_i^k + v_i^{k+1},\ i = 1, \cdots, \Omega, \quad (21)$$

where Ω is the swarm size, v_i^k and x_i^k represent the velocity and position of particle i at kth iteration step, respectively. r_1 and r_2 are two independent random numbers in the range of [0, 1], c_1 and c_2 are acceleration constants, usually $c_1 = c_2 = 2.0$. The parameter w is called the inertia weight factor. Generally, the value of each component in v_i can be limited to a range $[v_{min}, v_{max}]$ to control excessive roaming of particles outside the searching space. With (20) and (21), all particles in the swarm find their new positions and apply these new positions to update their individual best positions and global best position of the swarm. This process is repeated until a user-defined stopping criterion, usually maximum iteration number t_{max} is reached. Since the PSO is easy to implement and has various application areas, many researchers have conducted studies about it. For more details one can consult the references [6, 13].

3.2 Modification of PSO

The suitable velocity of particles is the most important factor in accuracy and convergence speed of PSO algorithm. High velocity causes wander particles and low velocity leads bars particles to move off global optimum points. PSO suffers from premature convergence which results in a low optimization precision or even failure when it is applied on high-dimensional problems. It should be noticed that high velocity leads to global search at initial iterations. Thus one does not fall to local optimal solutions. On the other hand, low velocity leads to more accurate solutions at eventual steps. Therefore, we consider high velocity in initial iteration and decrease velocity in any iteration until it approximately equal to zero in final iteration step. By considering relation (20) and (21), it is obvious that velocity of particle in each iteration depends on inertia weight. Yaun and et al consider the constant inertia weight as 0.7 [29]. Moreover Tripathi et al. [27] consider the inertia weight in the following form:

$$w_k = \left((w_1 - w_2) \frac{D - k}{D} \right) + w_2; \quad (22)$$

Where D is number of iteration, k is the number of current iteration and w_1, w_2 are given constants. Based on some numerical simulations, we believe that the following inertia weight is more appropriate in synchronization problem.

$$w = 0.7(1 - \frac{k-1}{D}); \quad (23)$$

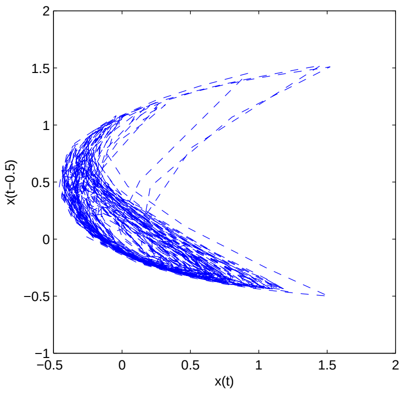

Fig. 1: Portrait strange attractor of logistic system

4 Numerical results

In this section, we consider the synchronization problem of logistic, Chen, Liu and Lorenz systems. We adopt modified PSO and $D = 150$ in Eq. (23).

4.1 Synchronization of delayed fractional-order logistic system

The fractional-order logistic delayed systems are rewritten by [30]:

$$\begin{cases} {}_LD^{\alpha}x_1(t) = -ax_1(t) + rx_1(t-\tau) + u(t) \\ {}_LD^{\alpha}x_2(t) = -ax_2(t) + rx_2(t-\tau) \end{cases} \tag{24}$$

where $a = 26$, $\alpha = 0.9$, $r = -53$, $\tau = 0.5$ and $u(t)$ is an unknown control function. In the absence of control function, there exists a chaotic attractor in system (24), as shown in Fig. 1. Consider initial conditions $x_1(t) = 0.5$ and $x_2(t) = 0.9$ for $t \in [-\tau, 0]$. Let $-50 \leq u_1(t) \leq 50$. We assume that the time step and final time are $h = 0.01$ and $T = 2$ respectively. The estimated result of synchronization is shown in Fig. **??**. The calculated control function is depicted in Fig. 3.

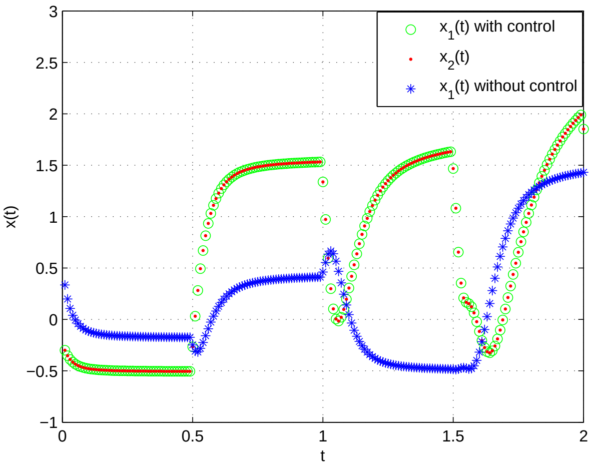

Fig. 2: Synchronization portrait of logistic system.
The green circles shows the behavior of $x_1(t)$. The trend of $x_2(t)$ is depicted by red points. The trend of $x_1(t)$ without control is shown by blue stars.

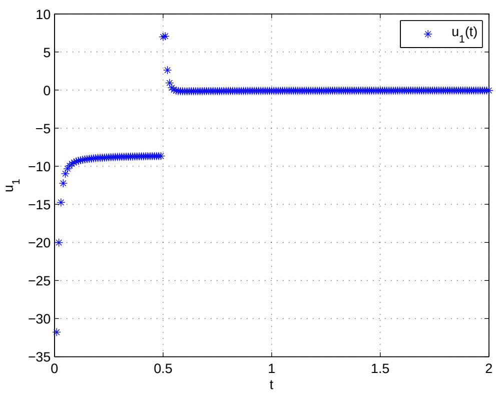

Fig. 3: Profile of optimal control function of logistic systems calculated by MPSO

4.2 Synchronization of fractional-order Chen system

The fractional-order Chen delayed system are rewritten by [29]:

$$\begin{cases} D^\alpha x_1(t) = a(y_1(t) - x_1(t-\tau)) + u_1(t) \\ D^\alpha y_1(t) = (c-a)x_1(t-\tau) - x_1(t)z_1(t) + cy_1(t)) + u_2(t) \\ D^\alpha z_1(t) = x_1(t)y_1(t) - bz_1(t-\tau) + u_3(t) \\ D^\alpha x_2(t) = a(y_2(t) - x_2(t-\tau)) \\ D^\alpha y_2(t) = (c-a)x_2(t-\tau) - x_2(t)z_2(t) + cy_2(t)) \\ D^\alpha z_2(t) = x_2(t)y_2(t) - bz_2(t-\tau) \end{cases} \tag{25}$$

where $a = 35$, $b = 3$, $c = 27$, $a = 0.94$, $\tau = 0.5$ and $u(t) = (u_1(t), u_2(t), u_3(t))$ is an unknown control function. In the absence of control function, there exists a chaotic attractor in system (25), as shown in Fig. 4. Consider initial conditions $x_1(t) = 0$, $y_1(t) = 0.5$, $z_1(t) = 0.2$, $x_2(t) = 0.2$, $y_2(t) = 0$, $z_2(t) = 0.5$ for $t \in [-\tau, 0]$ and $-300 \leq u_1(t), u_2(t), u_3(t) \leq 300$. We assume that the time step and final time are $h = 0.001$ and $T = 0.3$ respectively. The estimated result of synchronization is shown in Fig. 5. The calculated control function is depicted in Fig. 6.

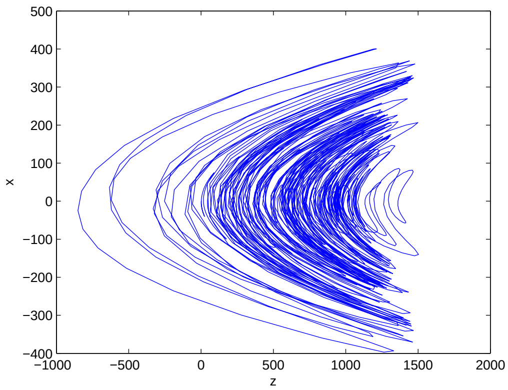

Fig. 4: Portrait strange attractor of Chen system

4.3 Synchronization of fractional-order Liu delayed system

The fractional-order Liu delayed systems are considered as follows [22,32]:

$$\begin{cases} D^{\alpha}x_1(t) = z_1(t) - ax_1(t) + x_1(t)y_1(t-\tau) + u_1(t) \\ D^{\alpha}y_1(t) = 1 - by_1(t) - x_1^2(t-\tau) + u_2(t) \\ D^{\alpha}z_1(t) = -x_1(t-\tau) - cz_1(t) + u_3(t) \\ D^{\alpha}x_2(t) = z_2(t) - ax_2(t) + x_2(t)y_2(t-\tau) \\ D^{\alpha}y_2(t) = 1 - by_2(t) - x_2^2(t-\tau) \\ D^{\alpha}z_2(t) = -x_2(t-\tau) - cz_2(t) \end{cases} \tag{26}$$

where $\alpha = 0.92$, $a = 3$, $b = 0.1$, $c = 1$, $\tau = 0.01$ and $u(t) = (u_1(t), u_2(t), u_3(t))$ is an unknown control function. In the absence of control function, there exists an chaotic attractors in system (26), as shown in Fig. 7. Consider initial conditions $x_1(t) = 0.1$, $y_1(t) = 4$, $z_1(t) = 0.5$, $x_2(t) = 2$, $y_2(t) = 0$, $z_2(t) = 0.7$ for $t \in [-\tau, 0]$ in (26). We assume that $-500 \leq u_1(t), u_2(t), u_3(t) \leq 500$ and $T = 0.04$ is final time. The time step is considered as $h = 0.002$. The estimated

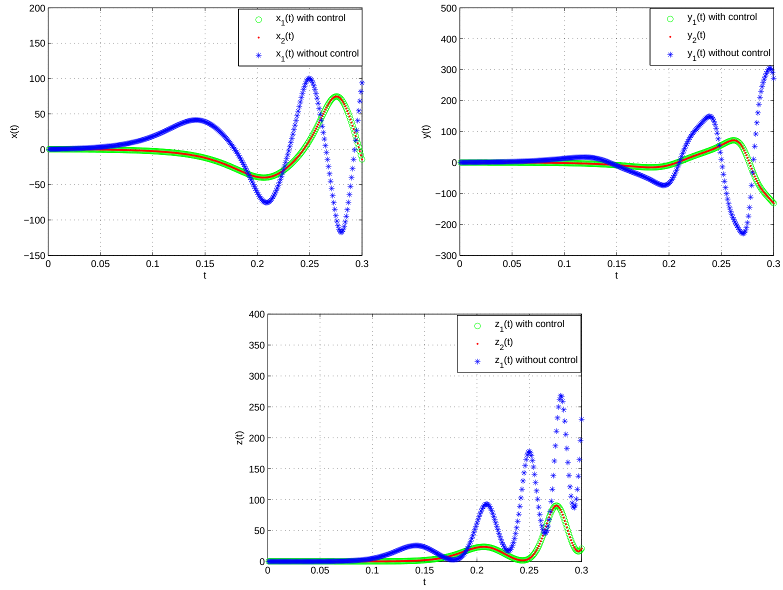

Fig. 5: Synchronization portrait of Chen system (25).
(a): The green circles shows the behavior of $x_1(t)$. The trend of $x_2(t)$ is depicted by red points. The trend of $x_1(t)$ without control is shown by blue stars.
(b) : The green circles shows the behavior of $y_1(t)$. The trend of $y_2(t)$ is depicted by red points. The trend of $y_1(t)$ without control is shown by blue stars.
(c) : The green circles shows the behavior of $z_1(t)$. The trend of $z_2(t)$ is depicted by red points. The trend of $z_1(t)$ without control is shown by blue stars.

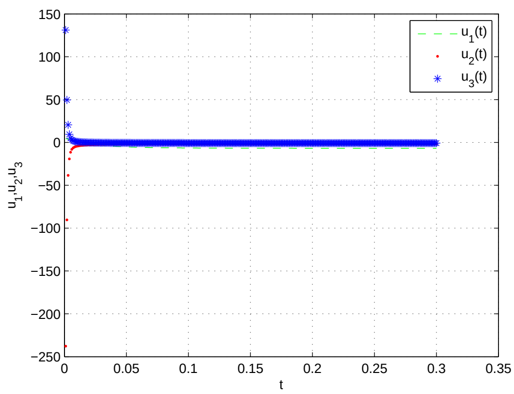

Fig. 6: Profile of optimal control function of Chen system calculated by MPSO

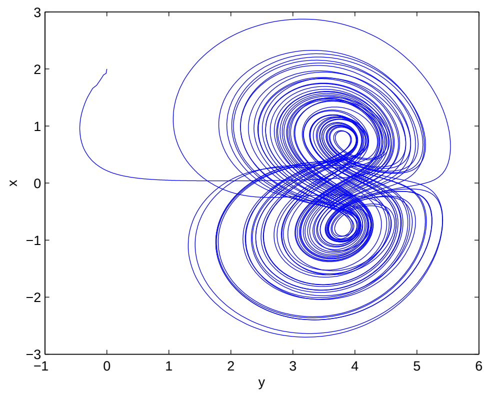

Fig. 7: Portrait strange attractor of Liu system

result of synchronization is shown in Fig. 8. The calculated control function is depicted if Fig. 9.

4.4 Lag Synchronization of fractional-order Liu delayed system

Consider the fractional-order Liu delayed system (26) and consider (18) as following:

$$\begin{aligned} &\min \ \sum_{i=1}^{7} |x(t_i) - y(t_{i+3})| + \sum_{i=8}^{N} |x(t_i) - y(t_{i-5})| \\ &s.t. \quad A_j \leq u_j(t_i) \leq B_j, \ i = 1, \cdots, N, \ j = 1, \cdots, \mathbb{k} \end{aligned} \tag{27}$$

$$\begin{aligned} &\min \ \sum_{i=1}^{7} |x(t_i) - y(t_{i+3})| + \sum_{i=8}^{N} |x(t_i) + y(t_{i-5})| \\ &s.t. \quad A_j \leq u_j(t_i) \leq B_j, \ i = 1, \cdots, N, \ j = 1, \cdots, \mathbb{k} \end{aligned} \tag{28}$$

where $-500 \leq u_j(t) \leq 500$ and $N = 20$. The Results are shown in Fig. 10 and Fig. 11.

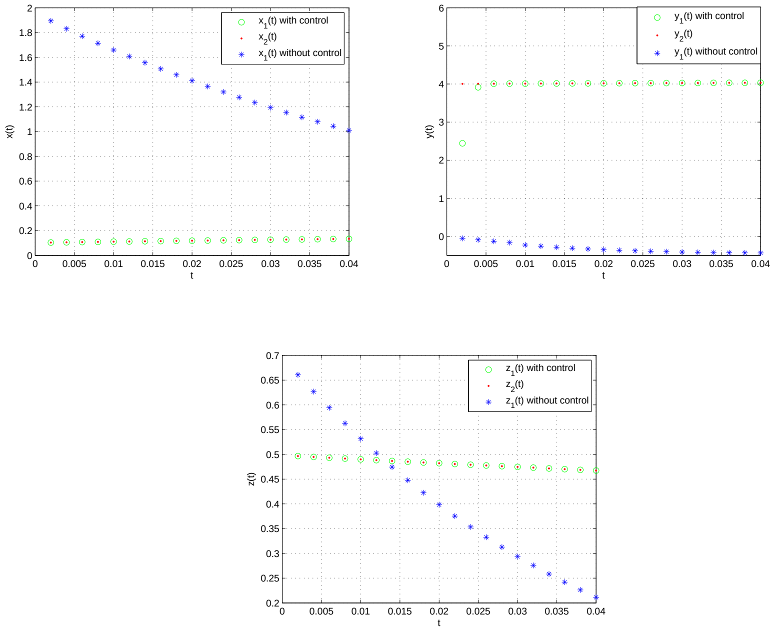

Fig. 8: Synchronization portrait of Liu system (26).
(a): The green circles shows the behavior of $x_1(t)$. The trend of $x_2(t)$ is depicted by red points. The trend of $x_1(t)$ without control is shown by blue stars.
(b) : The green circles shows the behavior of $y_1(t)$. The trend of $y_2(t)$ is depicted by red points. The trend of $y_1(t)$ without control is shown by blue stars.
(c) : The green circles shows the behavior of $z_1(t)$. The trend of $z_2(t)$ is depicted by red points. The trend of $z_1(t)$ without control is shown by blue stars.

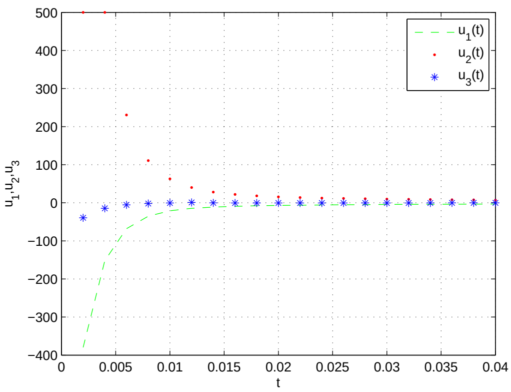

Fig. 9: Profile of optimal control function of Liu system calculated by MPSO

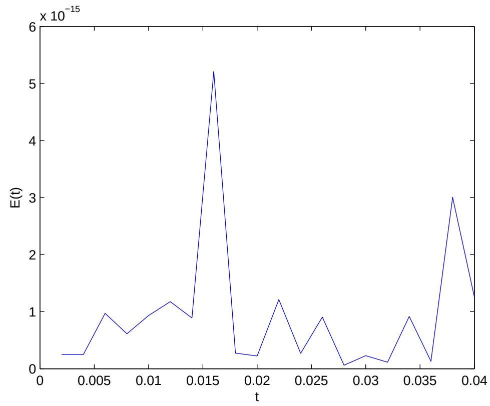

Fig. 10: The trend of error function in lag synchronization of delayed fractional-order Liu system defined in (27).

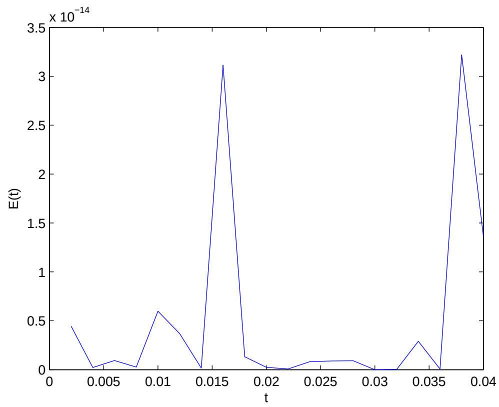

Fig. 11: The trend of error function in lag synchronization of delayed fractional-order Liu system defined in (28).

4.5 Synchronization of delayed fractional-order Chen system and Liu system

The fractional-order Chen delayed system and fractional-order Liu delayed system are rewritten by

$$\begin{cases} {}_LD^{\alpha}x_1(t) = a_1(y_1(t) - x_1(t-\tau_1)) + u_1(t) \\ {}_LD^{\alpha}y_1(t) = (c_1 - a_1)x_1(t-\tau_1) - x_1(t)z_1(t) \\ +c_1y_1(t) + u_2(t) \\ {}_LD^{\alpha}z_1(t) = x_1(t)y_1(t) - b_1z_1(t-\tau_1) + u_3(t) \\ D^{\beta}x_2(t) = z_2(t) - a_2x_2(t) + x_2(t)y_2(t-\tau_2) \\ D^{\beta}y_2(t) = 1 - b_2y_2(t) - x_2^2(t-\tau_2) \\ D^{\beta}z_2(t) = -x_2(t-\tau_2) - c_2z_2(t) \end{cases} \tag{29}$$

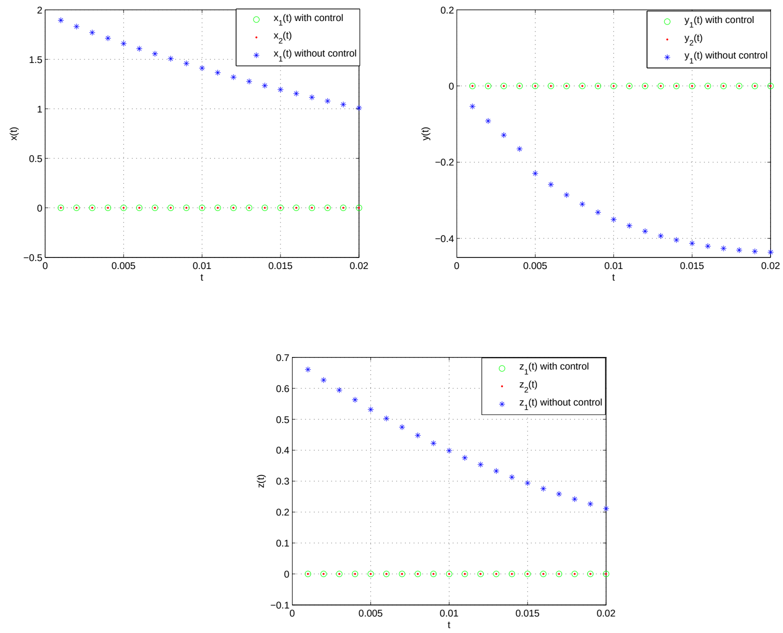

Fig. 12: Synchronization portrait of Liu system (26) and Chen system (29).
(a): The green circles shows the behavior of $x_1(t)$. The trend of $x_2(t)$ is depicted by red points. The trend of $x_1(t)$ without control is shown by blue stars.
(b) : The green circles shows the behavior of $y_1(t)$. The trend of $y_2(t)$ is depicted by red points. The trend of $y_1(t)$ without control is shown by blue stars.
(c) : The green circles shows the behavior of $z_1(t)$. The trend of $z_2(t)$ is depicted by red points. The trend of $z_1(t)$ without control is shown by blue stars.

where $a_1 = 35$, $b_1 = 3$, $c_1 = 27$, $\alpha = 0.94$, $\tau_1 = 0.5$, $\beta = 0.92$, $a_2 = 3$, $b_2 = 0.1$, $c_2 = 1$, $\tau_2 = 0.01$ Consider initial conditions $x_1(t) = 0$, $y_1(t) = 0.5$, $z_1(t) = 0.2$, $x_2(t) = 0.2$, $y_2(t) = 0$, $z_2(t) = 0.5$ for $t \in [-\tau_2, 0]$ and $-300 \leq u_1(t), u_2(t), u_3(t) \leq 300$. We assume that the time step and final time are $h = 0.001$ and $T = 0.02$ respectively. The estimated result of synchronization is shown in Fig. 12. The calculated control function is depicted in Fig. 13.

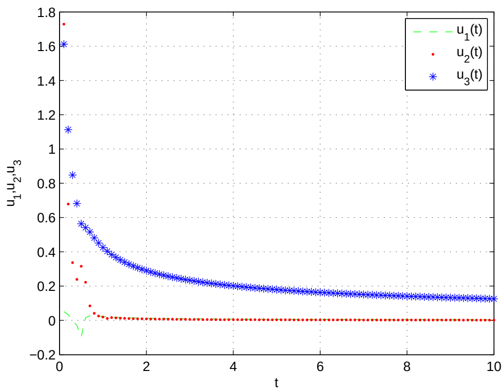

Fig. 13: Profile of optimal control function in synchronization between Liu and Chen systems calculated by MPSO

4.6 Synchronization of delayed fractional-order Lorenz system and Chen

The fractional-order Lorenz delayed system and fractional-order Chen delayed system are considered as follows

$$\begin{cases} D^{\alpha}x_1(t) = -y_1(t) - x_1(t-\tau) + u_1(t) \\ D^{\beta}y_1(t) = -x_1(t)z_1(t-\tau) + u_2(t) \\ D^{\gamma}z_1(t) = x_1(t)y_1(t-\tau) + R + u_3(t) \\ {}_LD^{q}x_2(t) = a(y_2(t) - x_2(t-\tau)) \\ {}_LD^{q}y_2(t) = (c-a)x_2(t-\tau) - x_2(t)z_2(t) + cz_2(t) \\ {}_LD^{q}z_2(t) = x_2(t)y_2(t) - bz_2(t-\tau) \end{cases} \tag{30}$$

where $R = 3.4693$, $\alpha = 0.8$, $\beta = 0.95$, $\gamma = 0.5$, $\tau = 0.5$, $q = 0.94$, $a = 35$, $b = 3$ and $c = 27$. In the absence of control function, there exist chaotic attractors in systems (30), as shown in Fig. 4 and Fig. 14. Consider initial conditions $x_1(t) = 0.2$, $y_1(t) = 0$, $z_1(t) = 0.5$, $x_2(t) = 0.2$, $y_2(t) = 0$, $z_2(t) = 0.5$ for $t \in [-\tau, 0]$. Let $-150 \leq u_1(t), u_2(t), u_3(t) \leq 150$. We assume that the time step and final time are $h = 0.1$ and $T = 35$ respectively. The estimated result of synchronization is shown in Fig. 15. The calculated control function is depicted in Fig. 16.

5 Conclusion

We have studied chaos synchronization of coupled fractional delay differential equations. The problem has been formulated as an infinite dimensional optimization problem. The infinite dimensional optimization problem is reduced to finite dimensional problem by discretizing the time interval. The modified

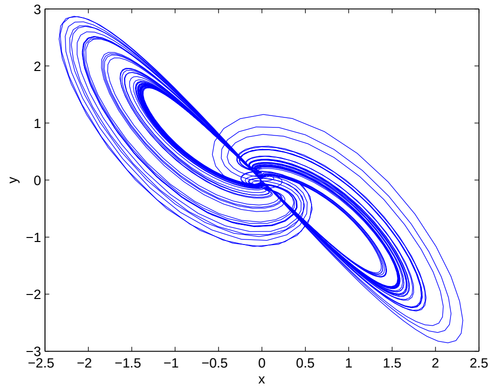

Fig. 14: Portrait strange attractor of Lorenz system

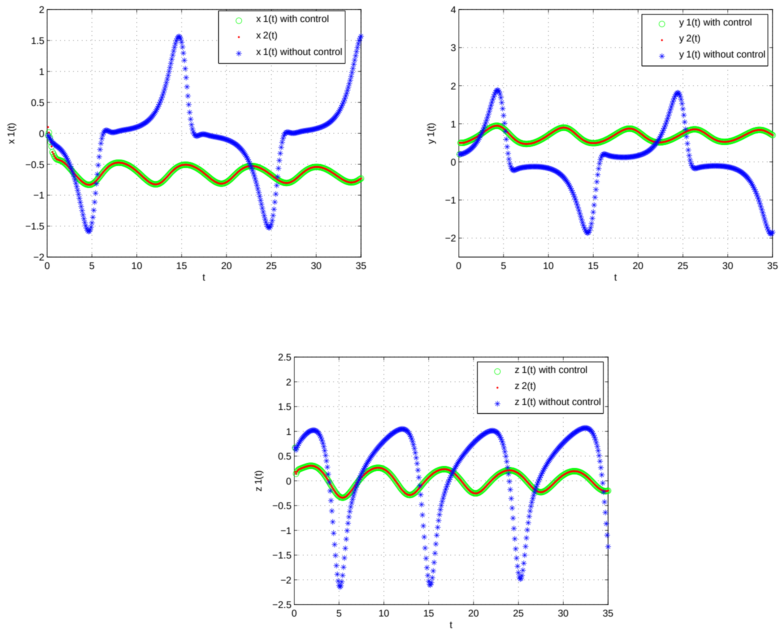

Fig. 15: Synchronization portrait of Lorenz system and Chen.
(a): The green circles shows the behavior of $x_1(t)$. The trend of $x_2(t)$ is depicted by red points. The trend of $x_1(t)$ without control is shown by blue stars.
(b) : The green circles shows the behavior of $y_1(t)$. The trend of $y_2(t)$ is depicted by red points. The trend of $y_1(t)$ without control is shown by blue stars.
(c) : The green circles shows the behavior of $z_1(t)$. The trend of $z_2(t)$ is depicted by red points. The trend of $z_1(t)$ without control is shown by blue stars.

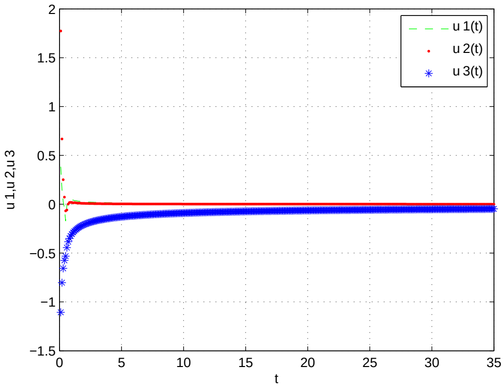

Fig. 16: Profile of optimal control function in synchronization between Lorenz and Chen systems calculated by MPSO

Table 1: Comparison of complete synchronization error and norm of controller between PSO and MPSO in Logistic, Chen, Lorenz and Liu systems

		PSO	MPSO
Complete synchronization Lorenz and Chen systems	E	$4.63606e-05$	$2.55458e-16$
	$\Vert U \Vert$	6.07709	6.07709
Complete synchronization Lorenz and Chen systems	E	$5.32625e-04$	$8.35580e-14$
	$\Vert U \Vert$	$8.65976e+02$	$8.65983e+02$
Complete synchronization Lorenz and Chen systems	E	$1.42721e-04$	$1.25698e-13$
	$\Vert U \Vert$	$2.96565e+02$	$2.96659e+02$
Complete synchronization Lorenz and Chen systems	E	$3.12659e-06$	$1.321656e-19$
	$\Vert U \Vert$	$7.43265e+01$	$7.43265e+01$
Complete synchronization Lorenz and Chen systems	E	$3.45469e-06$	$3.67629e-15$
	$\Vert U \Vert$	$4.66796e+02$	$4.66773e+02$

PSO algorithm is applied for solving the optimization problem. A comparison between PSO and MPSO is shown in Table 1, where the norm of local controller and the error of synchronization in N steps is defined as follow:

$$\|U\| = \left(\sum_{i=1}^{N}(u_1(t_i)^2 + u_2(t_i)^2 + u_3(t_i)^2)\right)^{\frac{1}{2}},$$

$$E = \left(\sum_{i=1}^{N}\left([x_1(t_i) - x_2(t_i)]^2 + [y_1(t_i) - y_2(t_i)]^2 + [z_1(t_i) - z_2(t_i)]^2\right)\right)^{\frac{1}{2}}.$$

Simulation results show that the MPSO method is clearly more efficient than PSO. Moreover, It is shown that the proposed method can be applied on wide range commensurate or non-commensurate master-slave systems. For further work, a suggested topic is applying MPSO for synchronization of coupled fractional-order delayed systems with random noise.

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