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Mittag-Leffler-Hyers-Ulam Stability of Fractional Differential Equation

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Abstract In this article, we study the Mittag-Leffler-Hyers-Ulam and Mittag-Leffler-Hyers-Ulam-Rassias stability of a class of fractional differential equation with boundary condition.

Keywords Fractional Differential Equation · Mittag-Laffler-Hyers-Ulam Stability

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1 Introduction

Consider fractional differential equation with the boundary condition

$$D^{\alpha}y(t) = F(t, y(t)), \tag{1}$$

$$ay(0) + by(T) = c. \tag{2}$$

Let Y be a normed space and $I = [0, T]$ be a given interval. Assume that for a continuously differentiable function $f : I \longrightarrow Y$ satisfying fractional differential inequality $\|^{c}D^{\alpha}f(t) - F(t, f(t))\| \leq \varepsilon$ for all $t \in I$ and for some $\varepsilon > 0$, where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, there exists a solution $f_0 : I \longrightarrow Y$ of the fractional boundary value problem (1) and (2) such that $\|f(t) - f_0(t)\| \leq K\varepsilon E_q$ for all $t \in I$. Then , we say that the above fractional boundary value problem (1) and (2) has the Mittag-Leffler-Hyers-Ulam stability. If the above statement is also true when we replace ε and $K\varepsilon$ by

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$\varphi(t)$ and $\Phi(t)$, where $\varphi, \Phi : I \longrightarrow [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Mittag-Leffler-Hyers-Ulam-Rassias stability. Fractional differential equations is the area of concentration of recent research and there has been significant progress in this area. However, the concept of fractional derivative is not new and is very much as old as differential equations. In 1695, L. Hospital raised the question about fractional derivative in a letter written to Leibniz and related his generalization of differentiation.
Recently, the differential equations of fractional order has proved to be a valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find many applications in electromagnetic, control, electrochemistry etc. For more details on this area, one can see the monographs of Kilbas et al. [9], Miller and Ross [10], I. Podulbny [17], Benchora [2] and the references therein.
In this paper, we will prove Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of fractional differential equation of (1) with the boundary condition (2).

2 Preliminaries

In this section, we give some basic definition and theorems which we used to prove the results.

Definition 1 For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \longrightarrow [0, +\infty]$ is called a generalized metric on X if and only if satisfies

(A_1) $d(x, y) = 0$ if and only if $x = y$;
(A_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(A_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The above concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space.

We now introduce one of the fundamental results of Banach fixed point theorem in a generalized complete metric space.

Theorem 1 *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \longrightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer K such thatd$(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:*

(a) *The sequence $\Lambda^n x$ convergence to a fixed point x^* of Λ;*
(b) *x^* is the unique fixed point of Λ in*

$$X^* = \{y \in X | d(\Lambda^k x, y) < \infty\};$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1 - L} d(\Lambda y, y).$$

Definition 2 [2] The fractional order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,$$

where Γ is the gamma function.

Definition 3 [2] For a function h given on the interval $[a,b]$, the α-th Riemann-Liouville fractional order derivative of h, is defined by

$$(D_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1}h(s)ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 4 [2] For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by

$$({}^cD_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 5 [2] A function $y \in C(j,\mathbb{R})$ is said to be a solution of (1)-(2) if y satisfies the equation ${}^cD^\alpha y(t) = f(t,y(t))$ on J, and the condition $ay(0) + by(T) = c$

Lemma 1 *[2] Let $0<\alpha<1$ and let $f:[0,T]\longrightarrow \mathbb{R}$ be continuous. A function $y \in C(J,\mathcal{R})$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds - \frac{1}{a+b}[\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s,y(s))ds - c]$$

if and only if y is a solution of the fractional boundary value problem

$$ {}^cD^\alpha y(t) = f(t,y(t)), t\in[0,T]$$
$$ay(0)+by(T)=c.$$

Theorem 2 *[19, Theorem 1] Suppose that $\hat{a}$ is a nonnegative function locally integrable on $[0,\infty)$ and $\hat{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\hat{g}(t)\le M, t\in[0,\infty)$, and suppose $u(t)$ is nonnegative and locally integrable on $[0,\infty)$ with*

$$u(t) \le \hat{a}(t) + \hat{g}(t)\int_0^t (t-s)^{q-1}u(s)ds, \quad t\in[0,\infty).$$

Then

$$u(t) \le \hat{a}(t)\int_0^t [\sum_{n=1}^{\infty}\frac{(\hat{g}(t)\gamma(q))^n}{\Gamma(nq)}(t-s)^{nq-1}\hat{a}(s)]ds, \quad t\in[0,\infty).$$

Remark 1 [19] Under the hypothesis of Theorem 2, let $\hat{a}(t)$ be a nondecreasing function on $[0,\infty)$. Then we have $u(t)\le \hat{a}(t)E_q[\hat{g}(t)\Gamma(q)t^q]$, where E_q is the Mittag-Leffler function defined by $E_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+1)}, \quad z\in\mathbb{C}.$

3 Mittag-Leffler-Hyers-Ulam-Rassias Stability

Here, we prove the Mittag-Leffler-Hyers-Ulam-Rassias Stability of the fractional differential equation (1) with the boundary condition (2) in the interval $[0, T]$ via Theorem 1.

Theorem 3 *Let $I = [0, T]$ be a closed interval. Let K, P, and L be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition*

$$|F(t, y) - F(t, z)| \leq L|y - z| \tag{3}$$

for any $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \longrightarrow \mathbb{R}$ satisfies

$$|{}^{c}D^{\alpha}y(t) - F(t, y(t))| \leq \varphi(t) \tag{4}$$

for all $t \in I$, where $\varphi : I \longrightarrow (0, \infty)$ is a continuous function with

$$\left|\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \varphi(\tau) d\tau\right| \leq K\varphi(t) \tag{5}$$

for all $t \in I$,

$$\left(\int_0^t (\varphi(\tau))^{\frac{1}{p}} d\tau\right)^p \leq M\varphi(t) \tag{6}$$

then, there exists unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y_0(s)) ds - \frac{b}{(a + b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y_0(s)) ds + \frac{c}{a + b} \tag{7}$$

and

$$|y(t) - y_0(t)| \leq \frac{M\varphi(t)E_{\alpha}(t)}{1 - KPL} \tag{8}$$

Proof Let us define a set X of all continuous functions $f : I \longrightarrow \mathbb{R}$ by

$$X = \{f : I \longrightarrow \mathbb{R} \mid f \quad is \quad a \quad continuous \quad function\} \tag{9}$$

similar to Theorem 3.1 of Jung S-M [7], we introduce a generalized complete metric on X as follows

$$d(f, g) = \inf\{C \in [0, \infty] \mid \quad |f(t) - g(t)| \leq C\varphi(t) \quad for \quad all \quad t \in I\}. \tag{10}$$

define an operator $\Lambda : X \longrightarrow X$ by

$$(\varLambda f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,f(s))ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,f(s))ds \tag{11}$$

$$+\frac{c}{a+b}$$

for all $f \in X$. It is easy to see that $\varLambda$ is well defined, since F and f are continuous functions.
To achieve our aim, we need to prove that $\varLambda$ is strictly contractive on X. For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \le C_{fg}$, that is by (10) we have

$$|f(t) - g(t)| \le C_{fg}\varphi(t) \tag{12}$$

for any $t \in I$. It then follows from (3), (5), (10), (11) and (12) that

$$|(\varLambda f)t - (\varLambda g)t| = |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[F(s,f(s)) - F(s,g(s))]ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}[F(s,f(s)) - F(s,g(s))]ds|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|F(s,f(s)) - F(s,g(s))|ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|F(s,f(s)) - F(s,g(s))|ds$$

$$\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s)-g(s)|ds - \frac{bL}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s)-g())|ds$$

$$\le \frac{L}{\Gamma(\alpha)}C_{fg}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds - \frac{bL}{(a+b)\Gamma(\alpha)}C_{fg}\int_0^T (T-s)^{\alpha-1}\varphi(s)ds$$

$$\le KPLC_{fg}\varphi(t)$$

for all $t \in I$. that is,

$$d(\varLambda f, \varLambda g) \le KLPC_{fg}.$$

Hence we can conclude that

$$d(\varLambda f, \varLambda g) \le KLPd(f,g)$$

for all $f, g \in X$. where we note that $0 < KLP < 1$. It follows from (9) and (11) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\varLambda g_0)(t) - g_0(t)| = |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,f(s))ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,f(s))ds+\frac{c}{a+b}-g_0(t)|\leq C\varphi(t)$$

for all $t\in I$ since $F(t,g_0(t))$ and $g_0(t)$ are bounded on I and $\min_{t\in I}\varphi(t)>0$. Thus (10) implies that

$$d(\Lambda g_0,g_0)<\infty \tag{13}$$

Therefore, Theorem 1 (a) implies that there exists a continuous function $y_0 : I\longrightarrow\mathbb{R}$ such that $\Lambda^n g_0\longrightarrow y_0$ in (X,d) as $n\longrightarrow\infty$ and such that $y_0=\Lambda y_0$ that is y_0 satisfies equation (6) for any $t\in I$. If $g\in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C_g>0$ with $|g_0(t)-g(t)|\leq C_g\varphi(t)$ for all $x\in I$.

This implies that $d(g_0,g)<\infty$ for every $g\in X$. or equivalently $\{g\in X\mid d(g_0,g)<\infty\}=X$. Therefore, according to Theorem 1 (b) y_0 is a unique continuous function with the property (7). Furthermore, it follows from (4) that

$$-\varphi(t)\leq {}^c D^{\alpha}_{a^+}y(t)-F(t,y(t))\leq\varphi(t), \tag{14}$$

for all $t\in I$. If we integrate each term of the above inequality and substitute the boundary conditions we obtain

$$|y(t)-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,y(s))ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,y(s))ds+\frac{c}{a+b}|\leq\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds$$

$$\leq\frac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}ds)^{1-p}(\int_0^t (\varphi(s))^{\frac{1}{p}}ds)^p$$

$$\leq\frac{1}{\Gamma(\alpha)}(\frac{t^{\frac{\alpha-1}{1-p}}}{\frac{\alpha-p}{1-p}})^{1-p}M\varphi(t)\leq M\varphi(t)E_\alpha(t)$$

for any $t\in I$.
Thus by (5) and (11) we get $|y(t)-(\Lambda y)(t)|\leq M\varphi(t)E_\alpha(t)$ for each $t\in I$, which implies that

$$d(y,\Lambda y)\leq M\varphi(t)E_\alpha(t) \tag{15}$$

Finally Theorem 1 (c) together with (15) implies that

$$d(y,y_0)\leq\frac{1}{1-LKP}d(\Lambda y,y)\leq\frac{M\varphi(t)E_\alpha(t)}{1-LKP}.$$

4 Mittag-Leffler-Hyers-Ulam Stability of the first type

In this section, we prove the Mittag-Leffler-Hyers-Ulam Stability of fractional differential equation (1) with the boundary condition (2).

Theorem 4 *Let $I = [0, T]$ be a closed interval and let $r > 0$ be a positive constant with $0 \leq t \leq r$. Let $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which satisfies the Lipschitz condition (3) for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < \dfrac{LPr^{\alpha}}{\Gamma(\alpha+1)} < 1$. If a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies the differential inequality*

$$|{}^{c}D^{\alpha}_{a+}y(t) - F(t, y(t))| \leq \varepsilon E_{\alpha}(t^{\alpha}) \tag{16}$$

for all $t \in I$, and for some $\varepsilon \geq 0$,

then there exists unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ satisfying equation (7) and

$$|y(t) - y_0(t)| \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1) - LPr^{\alpha}} \varepsilon E_{\alpha}(t^{\alpha}) \tag{17}$$

Proof First we define a set X of all continuous functions $f : I \longrightarrow \mathbb{R}$ by

$$X = \{f : I \longrightarrow \mathbb{R} \mid f \quad is \quad a \quad continuous \quad function\}$$

and introduce a generalized complete metric on X as follows

$$d(f, g) = \inf\{C \in [0, \infty] \mid \quad |f(t) - g(t)| \leq C \quad for \quad all \quad t \in I\}.$$

define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, f(s))ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, f(s))ds + \frac{c}{a+b} \tag{18}$$

for all $f \in X$. We now assert that Λ is strictly contractive on X. For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq C_{fg}$, that is, let us assume that

$$|f(t) - g(t)| \leq C_{fg} \tag{19}$$

for any $t \in I$. It then follows from (3), (18) and (19) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &= |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[F(s, f(s)) - F(s, g(s))]ds \\
&\quad - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}[F(s, f(s)) - F(s, g(s))]ds| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|F(s, f(s)) - F(s, g(s))|ds \\
&\quad - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|F(s, f(s)) - F(s, g(s))|ds
\end{aligned}$$

$$\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|f(s) - g(s)|ds - \frac{bL}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}|f(s) - g())|ds$$

$$\leq \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} ds - \frac{bL}{(a+b)\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} ds$$

$$\leq LC_{fg}[\frac{t^\alpha}{\alpha\Gamma(\alpha)} - \frac{bT^\alpha}{(a+b)\alpha\Gamma(\alpha)}]$$

$$\leq LC_{fg}[\frac{r^\alpha}{\alpha\Gamma(\alpha)} - \frac{br^\alpha}{(a+b)\alpha\Gamma(\alpha)}]$$

$$\leq \frac{LC_{fg}r^\alpha}{\Gamma(\alpha+1)}[\frac{a}{a+b}] \leq \frac{LPC_{fg}r^\alpha}{\Gamma(\alpha+1)}$$

for all $t \in I$. that is,

$$d(\Lambda f, \Lambda g) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)}.$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \leq \frac{LPr^\alpha}{\Gamma(\alpha+1)} d(f,g)$$

for all $f, g \in X$. where we note that $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$.
Analogously to the proof of Theorem 3, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$.
Therefore, Theorem 1 (a) implies that there exists a continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that $\Lambda^n g_0 \longrightarrow y_0$ in (X, d) as $n \longrightarrow \infty$ and such that $y_0 = \Lambda y_0$ that is y_0 satisfies equation (6) for any $t \in I$. If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with $|g_0(t) - g(t)| \leq C$ for all $t \in I$.

This implies that $d(g_0, g) < \infty$ for every $g \in X$. or equivalently $\{g \in X \mid d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 1 (b) y_0 is a unique continuous function with the property (7). Furthermore, it follows from (16) that

$$-\varepsilon E_\alpha(t^\alpha) \leq {}^c D^\alpha_{a^+} y(t) - F(t, y(t)) \leq \varepsilon E_\alpha(t^\alpha), \tag{20}$$

for all $t \in I$. If we integrate each term of the above inequality and substitute the boundary conditions, we obtain

$$|y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds + \frac{c}{a+b}| \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_\alpha(s^\alpha) ds$$

$$\leq \sum_{k=o}^{\infty} \frac{\varepsilon}{\Gamma(\alpha)\Gamma(k\alpha+1)} \int_0^t (t-s)^{\alpha-1} s^{k\alpha} ds$$

$$\leq \sum_{k=o}^{\infty} \frac{\varepsilon t^{(k+1)\alpha}}{\Gamma(\alpha)\Gamma(k\alpha+1)} \int_0^1 (1-s)^{\alpha-1} s^{k\alpha} ds$$

$$\leq \sum_{k=o}^{\infty} \frac{\varepsilon t^{(k+1)\alpha}}{\Gamma(\alpha)\Gamma(k\alpha+1)} \frac{\Gamma(\alpha)\Gamma(k\alpha+1)}{\Gamma(\alpha+(k\alpha+1))} \leq \varepsilon E_\alpha(t^\alpha)$$

for any $t \in I$. If we integrate each term of the above inequality and applying the boundary conditions, then we have

$$|(\Lambda y)(t) - y(t)| \leq \varepsilon E_\alpha(t^\alpha)$$

for any $t \in I$, that is, it holds that $d(y, \Lambda y) \leq \varepsilon E_\alpha(t^\alpha)$ It now follows from Theorem 1 (c) that

$$d(y, y_0) \leq \frac{1}{1 - \dfrac{LPr^\alpha}{\Gamma(\alpha+1)}} d(\Lambda y, y) \leq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1) - LPr^\alpha} \varepsilon E_\alpha(t^\alpha).$$

which implies the validity of (17) for each $t \in I$.

5 Mittag-Leffler-Hyers-Ulam Stability of the second type

Definition 6 Equation (1) is Mittag-Leffler-Hyers-Ulam stable of the second type, with respect to E_α, if for every $\varepsilon > 0$ and solution y of the following equation

$$|y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds + \frac{c}{a+b}| \tag{21}$$

$$\leq \frac{\varepsilon r^\alpha}{\Gamma(\alpha+1)}$$

there exists a solution $y_0 \in (C(I, \mathbb{R}))$ of equation (1) with

$$|y(t) - y_0(t)| \leq M E_\alpha(C t^\alpha)$$

for all $t \in I$ and $C \in \mathbb{R}$.

Theorem 5 *Let $I = [0, T]$ be a closed interval and let $r > 0$ be a positive constant with $0 \leq t \leq r$. Let $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which satisfies the Lipschitz condition 3 for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant that $M = 2\varepsilon E_\alpha(t^\alpha) - \dfrac{bLr^\alpha}{\Gamma(\alpha+1)(a+b)}$. Then equation (1) is Mittag-Leffler-Hyers-Ulam stable of the second order type.*

Proof Let $y \in C(I < \mathbb{R})$ satisfy the inequality (21). Let us denote by $y_0 \in C(I, \mathbb{R})$ the unique of solution to (1)-(2).
We have

$$|y(t) - y_0(t)| = |y(t) - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,y(s))ds + \frac{c}{a+b}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,y(s))ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,y(s))ds - \frac{c}{a+b}$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,y_0(s))ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,y_0(s))ds + \frac{c}{a+b}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s,y_0(s))ds$$

$$-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s,y_0(s))ds - \frac{c}{a+b} - y_0(t)|$$

$$\leq 2\varepsilon E_\alpha(t^\alpha) + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-y_0(s)|ds$$

$$-\frac{bL}{\Gamma(\alpha)(a+b)}\int_0^T (T-s)^{\alpha-1}|y(s)-y_0(s)|ds$$

$$\leq 2\varepsilon E_\alpha(t^\alpha) + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-y_0(s)|ds - \frac{bLr^\alpha}{\alpha\Gamma(\alpha)(a+b)}$$

$$\leq M + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-y_0(s)|ds$$

Now, by Remark 1, we have

$$u(t) \leq ME_\alpha(Lt^\alpha)$$

Thus, the conclusion of our theorem holds.

6 Conclusion

In this paper, we have discussed the Mittag-Leffler-Hyers-Ulam stability fractional differential equation with the boundary condition, that P. Muniyappan and S. Rajan [14] proved Hyers-Ulam stability of fractional differential equation (1) with the boundary condition (2).

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